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Response of Klystrons to Nanosecond Pulses

30 September 1962

N. Bose

LINEAR BEAM MICROWAVE TUBES, Technical Report No. 21 [Contract No. AF30(602)-2573]

School of Electrical Engineering
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Ithaca, New York

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CONTENTS

LIST OF SYMBOLS USED

α_n = $\frac{\text{magnitude of applied voltage}}{\text{d-c beam voltage}} = \frac{V_n}{V_o}$

δ_n = transit time correction factor for n^{th} gap

ω_n = angular frequency of applied voltages, V_n

ϕ''_m = d-c gap-transit angle for m^{th} gap and angular frequency ω_2

β''_m = gap-coupling coefficient for m^{th} gap and ω_2

k''_m = bunching parameter = $\frac{\omega_2 S_m}{u_o} \alpha_m \beta_m$

Γ_n = the n^{th} gap transit time

$\begin{bmatrix} A_0, A_1, \\ A_2, A_3, \end{bmatrix}$ = the four planes of reference in a two-cavity klystron

$\begin{bmatrix} a_o, a_r \\ b_r \end{bmatrix}$ = the Fourier coefficients

C = velocity of light

d_n = the length of n^{th} gridded gap

$e_{erf(x)}$ = error function(x) = $\frac{2}{\sqrt{\pi}} \int_o^x e^{-t^2} \, dt$

I_o = d-c beam current

i_n = total conduction current at plane A_n

J_p = Bessel function of first kind and p^{th} order

m = electronic mass

m_o = electronic mass at rest

R_e = real part

t = time that an electron arrives at an arbitrary plane Z within the gap

s_n = the n^{th} drift distance

S''_m = the normalized drift angle for the m^{th} drift space and angular frequency ω_2

t_n = time at which an electron passes plane A_n

u_o = d-c electron beam velocity

u_n = velocity of electron at an arbitrary plane Z in the n^{th} gap region

u = velocity of electron at an arbitrary plane Z in the first gap

v_1 = total electron velocity at exit from plane A

V_o = d-c electron beam velocity

V_n = amplitude of signal

z = arbitrary position co-ordinate of an electron within the gap

ABSTRACT

This investigation uses ballistic theory in the analysis of the behavior of an electron beam passing through alternate gap and drift regions, with the gap regions having excitation fields. The analysis develops the response to a complicated frequency spectrum of the drive signal. First the double frequency case is analyzed. Relativistic effects are taken into account and their influence on the current response studied. Then the theory is extended to the more complicated case of a Gaussian spectrum. The first-order bunching theory is used to plot current response curves. An estimate of the pulse distortion resulting from nonlinear electron beam dynamics is obtained from the curves. It is also of interest that the envelope shape of the exit current is almost completely independent of the r-f frequency. The large-signal, finite-gap analysis is carried out, and the results extended to the multiple-cavity klystron.

I. INTRODUCTION

A. OBJECTIVE

Many stages of development followed the invention of the klystron by the Varian brothers in 1939. Until now, most of the analysis and discussion has been limited to the case of a continuous-wave drive signal. The purpose here is to open the door to investigation of the numerous problems associated with the use of pulsed microwave amplifiers for amplification of nanosecond pulses whose pulse lengths are of the order of several cycles of the r-f carrier frequency. The main concern is with high-power amplifiers with average power capabilities comparable to conventional pulsed amplifiers; thus for comparable repetition rates, the peak power would be higher by the inverse ratio of the pulse lengths.

Several factors may be important in determining the pulse response capability of high-power amplifiers such as the klystron. It follows, from Fourier analysis, that a long pulse of constant carrier frequency includes a narrow bandwidth, while a pulse that is short in terms of cycles of the r-f carrier has a broad frequency spectrum. The spectrum of the long signal can, however, be significantly broadened by introducing modulation. Klauder[3] showed that to utilize the transmitting tubes efficiently, this modulation must take the form of frequency modulation. By this method one can introduce the frequency-spread characteristic of a short pulse within the envelope of a long-duration signal. Klauder[4] also showed certain advantages of short constant-frequency r-f signals over the long signals with linear frequency modulation. This emphasizes the importance of nanosecond pulse studies. One of the problems that arises is that the broad frequency spec-

trum associated with the short pulses might be affected by the bandwidth of the circuits associated with the amplifier, which will limit the response and therefore cause distortion of the pulse. Operation of the amplifier at maximum efficiency entails nonlinear behavior in the electron beam dynamics. This will cause the frequency spectrum of the output pulse to be altered from that of the input pulse and produce distortion. Ballistic theory will be utilized to determine the response of klystrons to the complicated frequency spectrum and lead to an estimate of the pulse distortion.

Studies in this direction will provide a solution for the conflicting requirements of long range and high resolution in radar systems. Resolution depends on the transmitted pulse bandwidth, and nanosecond pulses will, no doubt, satisfy the conditions for high resolution. For long-range capabilities, large power requirements are necessary; hence, high-power nanosecond pulses are expected to solve the two conflicting radar requirements. Radar systems that yield simultaneous information about the range and velocity of a target would be useful in certain applications. Klauder[4] showed an inherent ambiguity in a simultaneous determination of both the range and velocity of a moving target, when using the so-called "chirp" scheme. If the transmitted signal with an ambiguity function that is highly peaked only at about $t = 0$ exists, then high resolution is expected in both range and velocity. Using analogs from quantum mechanics, Klauder showed that the sequence of signals, $f(t)$, that satisfy these conditions are:

$$f(t) = \frac{\gamma^{1/4}}{\sqrt{\pi^{1/2} n! \; 2^n}} H_n(\sqrt{\gamma t}) \, e^{-\frac{\gamma t^2}{2}} ,$$

where $H_n(z)$ represents the n^{th} Hermite polynomial defined by

$$H_n(z) = (-1)^n e^{z^2} \frac{d^n}{dz^n} e^{-z^2} .$$

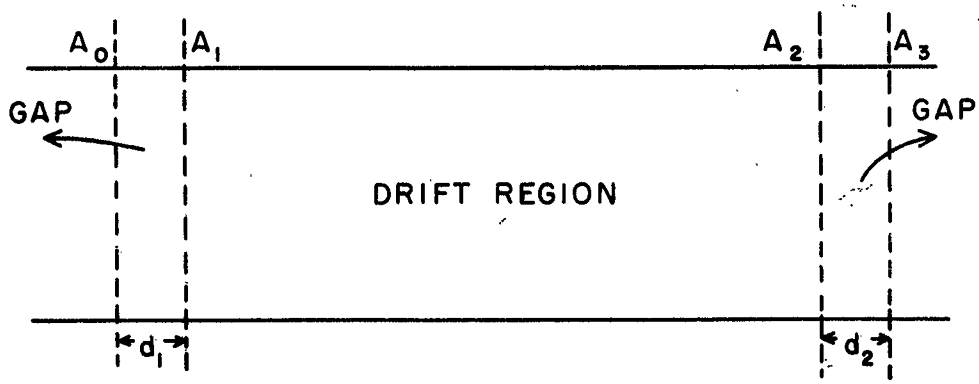
When n is taken equal to zero, the Gaussian envelope, on which this analysis is chiefly based, results. Further details on the Gaussian spectrum are given in Appendix A.

B. MODEL

Since a one-dimensional model of the electron beam is used, a univelocity electron beam is incident at the entrance plane $z = 0$, moving in the $+z$ direction in confined flow. This assumption of a very strong longitudinal magnetic field will depress the potential at the center of the beam so that peripheral electrons travel faster than axial ones, introducing a phase difference between the radio-frequency current carried by different beam segments. This difficulty is overcome by assuming the existence of a thread of positive ions along the axis of the beam, just sufficient to neutralize the charge density of electrons; thus, the effects of depressing the potential across the beam caused by space charge are neglected, and so also are variations in electron velocities caused by thermal noise. Electron velocities are assumed small compared to the velocity of light, permitting a nonrelativistic treatment of the problem. In the analysis of a double-frequency signal, however, the change in response caused by relativistic effects is studied. The electric field is assumed constant throughout the cross section of the klystron beam.

II. RESPONSE OF KLYSTRON TO DOUBLE-FREQUENCY INPUT

In this section, the response of a klystron to a double-frequency input will be treated. The signal is $V = V_1 \sin\omega_1 t + V_2 \sin\omega_2 t$. The



four planes of reference in a two-cavity klystron are represented by A_o, A_1, A_2, A_3, and the subscripts o, 1, 2, 3, respectively will be used to identify quantities in the respective planes.

A. FIRST GAP

Applying Newton's second law of motion, we have

$$\frac{d^2 z}{dt^2} = \frac{eV_1}{md_1} \sin\omega_1 t + \frac{eV_2}{md_1} \sin\omega_2 t \quad ;$$

therefore

$$\frac{dz}{dt} = \frac{-eV_1}{md_1\omega_1} \cos\omega_1 t - \frac{eV_2}{md_1\omega_2} \cos\omega_2 t + k_1 \quad ,$$

where k_1 is a constant to be determined from the initial condition; at $t = t_o$, the velocity $dz/dt = u_o$; therefore

$$\frac{dz}{dt} = u_o + \frac{eV_1}{md_1}\left[\frac{\cos\omega_1 t_o}{\omega_1} - \frac{\cos\omega_1 t}{\omega_1}\right] + \frac{eV_2}{md_1}\left[\frac{\cos\omega_2 t_o}{\omega_2} - \frac{\cos\omega_2 t}{\omega_2}\right] . \tag{2.1}$$

Integrating again gives

$$z = u_o t + \left[\frac{eV_1}{md_1}\,\frac{\cos\omega_1 t_o}{\omega_1} + \frac{eV_2}{md_1}\,\frac{\cos\omega_2 t_o}{\omega_2}\right] t$$

$$- \left[\frac{eV_1}{md_1\omega_1^2}\,\sin\omega_1 t + \frac{eV_2}{md_1\omega_2^2}\,\sin\omega_2 t\right] + k_2 ,$$

where k_2 is a constant, evaluated by inserting the initial condition: at $t = t_o$, the distance $z = 0$; therefore

$$z = \left[u_o + \frac{eV_1}{md_1}\,\frac{\cos\omega_1 t_o}{\omega_1} + \frac{eV_2}{md_1\omega_2}\,\cos\omega_2 t_o\right](t - t_o)$$

$$- \frac{eV_1}{md_1\omega_1^2}\,(\sin\omega_1 t - \sin\omega_1 t_o) - \frac{eV_2}{md_1\omega_2^2}\,(\sin\omega_2 t - \sin\omega_2 t_o) .$$

Putting

$$\frac{V_1}{V_o} = a_1 , \quad \frac{V_2}{V_o} = a_2 , \quad \frac{\omega_1 d_1}{u_o} = \phi_1' , \quad \frac{\omega_2 d_1}{u_o} = \phi_1'' ,$$

gives

$$z = \left[u_o + \frac{a_1 u_o}{2\phi_1'}\,\cos\omega_1 t_o + \frac{a_2 u_o}{2\phi_1''}\,\cos\omega_2 t_o\right](t - t_o) +$$

$$+ \frac{a_1 u_o}{2\phi_1' \omega_1} (\sin\omega_1 t_o - \sin\omega_1 t) + \frac{a_2 u_o}{2\phi_1'' \omega_2} (\sin\omega_2 t_o - \sin\omega_2 t) \quad . \quad (2.2)$$

The distance is $z = d$ at $t = t_1$, where

$$t_1 = t_o + \Gamma_1 ,$$

$$\Gamma_1 = \text{transit time in first gap} = \frac{d_1}{u_o} + \frac{\delta_1}{\omega_o} ,$$

$$\delta_1 = \text{correction factor},$$

$\omega_o = \dfrac{\omega_1 + \omega_2}{m + n}$, m and n being two numbers defined by $-\omega_1 = m\omega_o$ and $\omega_2 = n\omega_o$.

Substitution of this condition in Equation (2.2) gives,

$$d = u_o \left(1 + \frac{a_1}{2\phi_1'} \cos\omega_1 t_o + \frac{a_2}{2\phi_1''} \cos\omega_2 t_o\right) \Gamma_1$$

$$+ \frac{a_1 u_o}{2\phi_1' \omega_1} (\sin\omega_1 t_o - \sin\omega_1 t) + \frac{a_2 u_o}{2\phi_1'' \omega_2} (\sin\omega_2 t_o - \sin\omega_2 t) \quad . \quad (2.3)$$

The following assumptions are made:

$$\begin{matrix}\sin\\ \cos\end{matrix} \left(\omega_1 t - \phi_1' - \frac{\omega_1}{\omega_o} \delta_1\right) = \begin{matrix}\sin\\ \cos\end{matrix} \left(\omega_1 t - \phi_1'\right)$$

$$\begin{matrix}\sin\\ \cos\end{matrix} \left(\omega_2 t - \phi_1'' - \frac{\omega_2}{\omega_o} \delta_1\right) = \begin{matrix}\sin\\ \cos\end{matrix} \left(\omega_2 t - \phi_1''\right) \quad .$$

The products $\alpha_1 \delta_1$ and $\alpha_2 \delta_1$ are negligible, and Equation (2.3) reduces to,

$$- \frac{u_o \delta_1}{\omega_o} = \frac{\alpha_1 d_1}{2\phi_1'} \cos(\omega_1 t - \phi_1') + \frac{\alpha_2 d_1}{2\phi_1''} \cos(\omega_2 t - \phi_1'') + \frac{\alpha_1 u_o}{2\phi_1' \omega_1} \left[\sin(\omega_1 t - \phi_1') - \sin\omega_1 t\right]$$

$$+ \frac{\alpha_2 u_o}{2\phi_1'' \omega_2} \left[\sin(\omega_2 t - \phi_1'') - \sin\omega_2 t\right]$$

$$= \frac{\alpha_1 d_1}{2\phi_1'} (\cos\omega_1 t \cos\phi_1' + \sin\omega_1 t \sin\phi_1') + \frac{\alpha_2 d_1}{2\phi_1''} (\cos\omega_2 t \cos\phi_1'' + \sin\omega_2 t \sin\phi_1'')$$

$$+ \frac{\alpha_1 u_o}{2\phi_1' \omega_1} \left[\sin\omega_1 t \cos\phi_1' - \cos\omega_1 t \sin\phi_1' - \sin\omega_1 t\right]$$

$$+ \frac{\alpha_2 u_o}{2\phi_1'' \omega_2} \left[\sin\omega_2 t \cos\phi_1'' - \cos\omega_2 t \sin\phi_1'' - \sin\omega_2 t\right] ,$$

therefore

$$\delta_1 = \frac{\alpha_1}{2\phi_1' m} \left[(1 - \cos\phi_1' - \phi_1' \sin\phi_1') \sin\omega_1 t + (\sin\phi_1' - \phi_1' \cos\phi_1') \cos\omega_1 t\right]$$

$$+ \frac{\alpha_2}{2\phi_1'' n} \left[(1 - \cos\phi_1'' - \phi_1'' \sin\phi_1'') \sin\omega_2 t + (\sin\phi_1'' - \phi_1'' \cos\phi_1'') \cos\omega_2 t\right] . \tag{2.4}$$

Thus, the expression for the correction factor as given by Equation (2.4) is a superposition for each frequency, considered separately. The induced current is calculated by Ramo's theorem; i.e.

$$i_1 = \frac{I_o}{d_1} \int_{t-\Gamma_1}^{t} \left(\frac{dz}{dt}\right) dt_o = \frac{I_o}{d_1} \int_{t-J_1}^{t} \left\{ u_o + \frac{eV_1}{md_1} \left(\frac{\cos\omega_1 t_o}{\omega_1} - \frac{\cos\omega_1 t}{\omega_1} \right) \right.$$

$$+ \frac{eV_2}{md_1}\left(\frac{\cos\omega_2 t_o}{\omega_2} - \frac{\cos\omega_2 t}{\omega_2}\right)\Bigg\}\, dt_o$$

$$= \frac{I_o}{d_1}\left\{u_o\Gamma_1 + \frac{{}_1u_o}{2\phi_1'}\left[\frac{\sin\omega_1 t - \sin\omega_1(t-\Gamma_1)}{\omega_1} - \Gamma_1\cos\omega_1 t\right]\right.$$

$$\left. + \frac{{}_2u_o}{2\phi_1''}\left[\frac{\sin\omega_2 t - \sin\omega_2(t-\Gamma_1)}{\omega_2} - \Gamma_1\cos\omega_2 t\right]\right\}$$

After simplifying, and in the process neglecting $m\delta_1 \cos\omega_2 t$ and $n\delta_1 \cos\omega_1 t$, we have

$$i_1 = I_o + I_o\,a_1\left[\left(\frac{1-\cos\phi_1'}{2\phi_1'^2} - \frac{\sin\phi_1'}{2\phi_1'}\right)\sin\omega_1 t + \left(\frac{\sin\phi_1''}{\phi_1'^2} - \frac{1+\cos\phi_1'}{2\phi_1'}\right)\cos\omega_1 t\right]$$

$$+ I_o\,a_2\left[\left(\frac{1-\cos\phi_1''}{2\phi_1''^2} - \frac{\sin\phi_1''}{2\phi_1''}\right)\sin\omega_2 t + \left(\frac{\sin\phi_1''}{\phi_1''^2} - \frac{1+\cos\phi_1''}{2\phi_1''}\right)\cos\omega_2 t\right] \quad . \tag{2.5}$$

From Equation (2.5), it is evident that the effects of the different frequency components of excitation on the induced current are mutually independent. The double-frequency case can therefore be extended to the multiple-frequency input, and it can be concluded that the different frequency effects are independent of each other, subject to the approximations made in this section.

B. DRIFT REGION

Thus the electrons enter the drift space with both velocity and current modulation. From Equation (2.1),

$$u_1 = u_o + \frac{e V_1}{md_1\omega_1}\left[\cos\omega\left(t_1 - \frac{d_1}{u_o} - \frac{\delta_1}{\omega_o}\right) - \cos\omega_1 t_1\right] + \frac{e V_2}{md_1\omega_2}\left[\cos\omega_2\left(t_1 - \frac{d_1}{u_o} - \frac{\delta_1}{\omega_o}\right) - \cos\omega_2 t_1\right] .$$

Resorting to approximation made in Section A gives

$$u_1 = u_o + \frac{1}{2}\ u_o a_1 \frac{\sin\frac{\phi_1'}{2}}{\frac{\phi_1'}{2}}\ \sin\left(\omega_1 t_1 - \frac{\phi_1'}{2}\right) + \frac{1}{2}\ u_o a_2 \frac{\sin\frac{\phi_1''}{2}}{\frac{\phi_1''}{2}}\ \sin\left(\omega_2 t_1 - \frac{\phi_1''}{2}\right) . \tag{2.6}$$

Putting

$$\frac{\sin\frac{\phi_1}{2}}{\frac{\phi_1'}{2}} = \beta_1' , \qquad \frac{\sin\frac{\phi_1''}{2}}{\frac{\phi_1''}{2}} = \beta_1'' ,$$

where β_1' and β_1'' are defined as the gap-coupling coefficients for the respective frequency components of excitation, gives the time of arrival at plane A_2 as

$$t_2 = t_1 + \frac{S_1}{u_o\left[1 + \frac{1}{2}\ a_1\beta_1'\ \sin\left(\omega_1 t_1 - \frac{\phi_1'}{2}\right) + \frac{1}{2}\ a_2\beta_1''\ \sin\left(\omega_2 t_1 - \frac{\phi_1''}{2}\right)\right]} .$$

For small excitations, we get

$$t_2 = t_1 + \frac{S_1}{u_o}\left[1 - \frac{1}{2}\ a_1\beta_1'\ \sin\left(\omega_1 t_1 - \frac{\phi_1'}{2}\right) - \frac{1}{2}\ a_2\beta_1''\ \sin\left(\omega_2 t_1 - \frac{\phi_1''}{2}\right)\right]$$

$$\frac{dt_2}{dt_1} = 1 + \frac{S_1}{u_o} \left[- \frac{\omega_1 \alpha_1 \beta_1'}{2} \cos\left(\omega_1 t_1 - \frac{\phi_1'}{2}\right) - \frac{\omega_1 \alpha_2 \beta_1''}{2} \cos\left(\omega_2 t_1 - \frac{\phi_1''}{2}\right)\right] .$$

The equation for conservation of charge gives,

$$i_2 = \frac{i_1}{\frac{dt_2}{dt_1}} = \frac{1}{\frac{dt_2}{dt_1}} \frac{I_o}{\frac{dt_1}{dt_o}} . \tag{2.7}$$

Again $t_1 = t_o + \Gamma_1 = t_o + \frac{d_1}{u_o} + \frac{\delta_1}{\omega_o}$; therefore

$$\frac{dt_1}{dt_o} = 1 + \frac{d}{dt_o}\left(\frac{\delta_1}{\omega_o}\right) ,$$

as d_1 is independent of t_o, and

$$\frac{dt_1}{dt_o} = 1 + \frac{\alpha_1}{2\phi_1'}\left[\left(1 - \phi_1' \sin\phi_1' - \cos\phi_1'\right)\cos\left(\omega_1 t_o + \phi_1'\right) - \left(\sin\phi_1' - \phi_1' \cos\phi_1'\right)\sin\left(\omega_1 t_o + \phi_1'\right)\right]$$

$$+ \frac{\alpha_2}{2\phi_1''}\left[\left(1 - \phi_1'' \sin\phi_1'' - \cos\phi_1''\right)\cos\left(\omega_2 t_o + \phi_1''\right) - \left(\sin\phi_1'' - \phi_1'' \cos\phi_1''\right)\sin\left(\omega_2 t_o + \phi_1''\right)\right].$$

As a result,

$$i_2 = I_o \left(\left[1 - \frac{S_1' \alpha_1 \beta_1'}{2} \cos\left(\omega_1 t_1 - \frac{\phi_1'}{2}\right) - \frac{S_1'' \alpha_2 \beta_1''}{2} \cos\left(\omega_2 t_1 - \frac{\phi_1''}{2}\right)\right]\right.$$

$$\left\{1 + \frac{\alpha_1}{2\phi_1'}\left[\left(1 - \phi_1' \sin\phi_1' - \cos\phi_1'\right)\cos\left(\omega_1 t_o + \phi_1'\right) - \left(\sin\phi_1' - \phi_1' \cos\phi_1'\right)\sin\left(\omega_1 t_o + \phi_1''\right)\right]\right.$$

$$+ \frac{a_2}{2\phi_1''} \left[\left(1 - \phi_1'' \sin\phi_1'' - \cos\phi_1''\right) \cos\left(\omega_2 t_o + \phi_1''\right) - \left(\sin\phi_1'' - \phi_1'' \cos\phi_1''\right) \sin\left(\omega_2 t_o + \phi_1''\right)\right]\Bigg\}\Bigg)^{-1} \tag{2.8}$$

where

$$S_1' = \frac{\omega_1 S_1}{u_o} , \qquad S_1'' = \frac{\omega_2 S_1}{u_o} .$$

With

$$\begin{matrix}\sin\\ \cos\end{matrix} \left(\omega_1 t_o + \phi_1'\right) = \begin{matrix}\sin\\ \cos\end{matrix} \left(\omega_1 t_1\right) ,$$

and with approximations similar to those made previously, the current i_2 can be expressed as a function of t_1, and hence t_2.

The periodicity of i_2 is obvious, and therefore i_2 can be expanded in Fourier series as follows. First, t_2 will be related to t_o, as follows:

$$t_2 = t_o + \frac{d_1}{u_o} + \frac{S_1'}{u_o} \left[1 - \frac{1}{2} a_1\beta_1' \sin\left(\omega_1 t_o + \frac{\phi_1'}{2}\right) - \frac{1}{2} a_2\beta_1'' \sin\left(\omega_2 t_o + \frac{\phi_1''}{2}\right)\right] ,$$

i.e., the term δ_1/ω_o is neglected, while the approximations with regard to the sinusoidal terms are justified; then

$$i_2 = \frac{1}{2} a_o + \sum_{r=1}^{\infty} a_r \cos n\left(\omega t_2 - S_1 - \phi_1\right) + \sum_{r=1}^{\infty} b_r \cos n\left(\omega t_2 - S_1 - \phi_1\right) ,$$

where

$$S_1 = \frac{\omega_o S_1}{u_o} , \qquad \phi_1 = \frac{\omega_o d_1}{u_o} , \qquad a_o = 2 I_o ,$$

and

$$a_r = \frac{1}{\pi} \int_{-\pi}^{\pi} \frac{I_o}{\frac{dt_2}{dt_o}} \cos r(\omega t_2 - S_1 - \phi_1) \quad d(\omega t_2)$$

$$= \frac{1}{\pi} \int_{-\pi}^{\pi} I_o \cos r\left[\omega_o t_o - k_1' \sin\left(m\omega_o t_o + \frac{\phi_1'}{2}\right) - k_1'' \sin\left(n\omega_o t_o + \frac{\phi_1''}{2}\right)\right] d(\omega_o t_o),$$

$$b_r = \frac{1}{\pi} \int_{-\pi}^{\pi} I_o \sin r\left[\omega_o t_o - k_1' \sin\left(m\omega_o t_o + \frac{\phi_1'}{2}\right) - k_1'' \sin\left(n\omega_o t_o + \frac{\phi_1''}{2}\right)\right] d(\omega_o t_o), \tag{2.9}$$

where

$$k_1' = \frac{S_1 \alpha_1 \beta_1'}{2}, \qquad k_1'' = \frac{S_1 \alpha_1 \beta_1''}{2}.$$

The coefficients a_r and b_r are simplified by a method indicated in Appendix B.

The general equation of motion has to be formulated for the calculation of the induced current in plane A_3, and the method adopted is similar to that used in the first gap, with only the initial conditions different. This is done later for the more complicated Gaussian spectrum and will be omitted here. The main purpose of this section is to show the absence of intermodulation of the different frequency components at the output, subject, of course, to the approximations made. It must be stated that the preceding analysis was based upon frequencies ω_1 and ω_2 not being very far from the central frequency ω_o in the frequency spectrum, i.e., the numbers m and n should not be much greater than 1. As the main purpose of this study is the extension of this analysis to the response of a klystron to short pulses

with a narrow bandwidth of frequencies, the assumptions made are compatible with the condition desired.

No account has been taken of the relativistic variation of mass with velocities. This problem becomes especially serious when the beam voltage is large in high-power klystrons and where the very hard X-rays produced present an additional hazard to the operating personnel. A simple treatment will be given of the relativistic effects on the response, using the same model as before.

According to Einstein, nothing can move with a speed greater than the speed of light. Newtonian mechanics combined with this postulate demands that a mass subjected to a constant force must be accelerated till the speed of light is attained; but, as the force is still present, the speed must still increase, which is impossible. This ambiguity is solved by accepting the increase of mass with velocity, and assuming that mass is a manifestation of energy, the two related to each other by the famous equation $w = c^2 m$, where c is the velocity of light.

An increase in mass, dm, when accelerated, results in $c^2\, dm = d\omega = F\, ds$, where F is the applied force over a distance ds. Newton's, second law gives

$$F = \frac{d}{dt}\,(mv) \quad ;$$

therefore

$$c^2 \int dm = \int \frac{d}{dt}\,(mv)\; ds = \int v\, d\,(mv) \quad .$$

Equating the integrands and separating variables, we have

$$\frac{dm}{m} = \frac{v\, dv}{(c^2 - v^2)} \quad .$$

Assuming that rest mass equals m_o gives, by integration,

$$m = \frac{m_o}{\sqrt{1 - \frac{v^2}{c^2}}} \quad , \tag{2.10}$$

the equation that demonstrates the variation of mass with velocity. In this case, Newton's force equation gives

$$\frac{d}{dt}\left[\frac{m_o u}{\sqrt{1 - \frac{u^2}{c^2}}}\right] = eE \quad .$$

With $u(z,t) = u_o(z) + v(z)\, e^{j\omega t}$, where

$u(z, t)$ = total electron velocity,

$u_o(z)$ = d-c beam velocity,

$v(z)$ = amplitude of a-c velocity.

The basic assumption will be that $v \ll c$, which is justified, since the signal voltage is not sufficiently high in practice to make the a-c velocity appreciable in comparison to the speed of light. Using the Taylor series expansion, we have

$$\frac{u}{\left(1 - \frac{u^2}{c^2}\right)} = \frac{u_o}{\left(1 - \frac{u_o^2}{c^2}\right)^{1/2}} + ve^{j\omega t}\,\frac{d}{du_o}\left[\frac{u_o}{\left(1 - \frac{u_o^2}{c^2}\right)^{1/2}}\right] = \frac{u_o}{\left(1 - \frac{u_o^2}{c^2}\right)^{1/2}} + \frac{ve^{j\omega t}}{\left(1 - \frac{u_o^2}{c^2}\right)^{3/2}} \quad ;$$

therefore

$$\frac{d}{dt}\left[\frac{u_o}{\left(1-\frac{u_o^2}{c^2}\right)^{1/2}}+\frac{ve^{j\omega t}}{\left(1-\frac{u_o^2}{c^2}\right)^{3/2}}\right]=-\frac{e}{m_o}\,E\quad . \tag{2.11}$$

As the excitation is variational, separating Equation (2.11) into the d-c and a-c parts gives,

$$\frac{d}{dt}\left[\frac{u_o}{\left(1-\frac{u_o^2}{c^2}\right)^{1/2}}\right]=0\quad ,$$

$$\frac{d}{dt}\left[\frac{ve^{j\omega t}}{\left(1-\frac{u_o^2}{c^2}\right)^{3/2}}\right]=\frac{e}{m_o d_1}\left[V_1\ \sin\omega_1 t+V_2\ \sin\omega_2 t\right]\quad ; \tag{2.12a}$$

therefore

$$v_{1a\text{-}c}=-\left(1-\frac{u_o^2}{c^2}\right)^{3/2}\frac{e}{m_o d_1}\left[\frac{V_1\ \cos\omega_1 t}{\omega_1}+\frac{V_2\ \cos\omega_2 t}{\omega_2}\right]+k_3\quad ,$$

where k_3 is a constant evaluated from the initial condition that at $t = t_o$, $v_{1\,a\text{-}c} = 0$. Finally

$$v_{1\,a\text{-}c}=-\left(1-\frac{u_o^2}{c^2}\right)^{3/2}\frac{e}{m_o d_1}\left[V_1\left(\frac{\cos\omega_1 t_o}{\omega_1}-\frac{\cos\omega_1 t}{\omega_1}\right)+V_2\left(\frac{\cos\omega_2 t_o}{\omega_2}-\frac{\cos\omega_2 t}{\omega_2}\right)\right]\quad .$$

The total velocity at any instant in the gap is

$$\frac{dz}{dt} = u_o + v_{1\,a-c} \quad .$$

Integrating again and using the initial condition that $z = 0$ at $t = t_o$, we have

$$z = \left[u_o + \left(1 - \frac{u_o^2}{c^2}\right)^{3/2} \frac{e}{m_o d_1} \left(\frac{V_1 \cos\omega_1 t_o}{\omega_1} + \frac{V_2 \cos\omega_2 t_o}{\omega_2}\right)\right] (t - t_o)$$

$$+ \left(1 - \frac{u_o^2}{c^2}\right)^{3/2} \frac{e}{m_o d_1} \left(\frac{V_1 \sin\omega_1 t_o}{\omega_1^2} - \frac{V_1 \sin\omega_1 t_o}{\omega_1^2} + \frac{V_2 \sin\omega_2 t_o}{\omega_2^2} - \frac{V_2 \sin\omega_2 t_o}{\omega_2^2}\right) .$$

Now $z = d$ at $t = t_1 = t_o + \Gamma_1$. Following the procedure of Section A of this chapter, and making similar approximations, we obtain the expression for the correction factor in this equation:

$$\delta_1 = \frac{a_1 \omega_1}{2\phi_1' \omega_1} \left\{\left[\left(1 - \cos\phi_1' - \phi_1' \sin\phi_1'\right) \sin\omega_1 t + \left(\sin\phi_1' - \phi_1' \cos\phi_1'\right) \cos\omega_1 t\right]\right.$$

$$\left. + \frac{a_2 \omega_o}{2\phi_1'' \omega_2} \left[1 - \cos\phi_1'' - \phi_1'' \sin\phi_1'' \sin\omega_1 t + \left(\sin\phi_1'' - \phi_1'' \cos\phi_1''\right) \cos\omega_1 t\right]\right\}$$

$$\left(1 - \frac{u_o^2}{c^2}\right)^{3/2} \quad . \qquad (2.12b)$$

Equation (2.12b) is very similar to Equation (2.4) and shows that the correction factor is only multiplied by a constant $\left(1 - u_o^2\right)^{3/2} / c^2$ when relativistic effects are included. Again, the total current induced as a result of

the passage of electrons in the interval $t_o = t - \Gamma_1$, and $t_o = t$ is

$$i_1 = \frac{I_o}{d} \int_{t-\Gamma_1}^{t} u \cdot dt_o \quad .$$

Carrying out this integration as in the nonrelativistic case, we have

$$i_1 = I_o + \left(1 - \frac{u_o^2}{c^2}\right)^{3/2} I_o \left(a_1 \left\{\left(1 - \frac{\cos\phi_1'}{2\phi_1'^2} - \frac{\sin\phi_1'}{2\phi_1'}\right)\sin\omega_1 t + \left[\frac{\sin\phi_1'}{2\phi_1'^2} - \frac{(1+\cos\phi_1')}{2\phi_1'}\right]\cos\omega_1 t\right\}\right.$$
$$\left. + a_2\left\{\left(1 - \frac{\cos\phi_1'}{2\phi_1''^2} - \frac{\sin\phi_1''}{2\phi_1''}\right)\sin\omega_2 t + \left(\frac{\sin\phi_1''}{2\phi_1''^2} - \frac{1+\cos\phi_1''}{2\phi_1''}\right)\cos\omega_2 t\right\}\right) \quad . \tag{2.13}$$

From Equation (2.13) the relativistic effects on the induced current in the first gap are very clearly observed. The explicit effect on the r-f current resulting from electrons at high beam voltages subject to the simple approximations made should be noted. For a particular beam voltage, the r-f induced current is lower by the factor $\left(1 - u_o^2/c^2\right)^{3/2}$ when the relativistic variation of mass with velocity is taken into account. A curve has been plotted to show this effect (Figure 2).

For low voltages and hence low values of u_o, the factor $\left(1 - u_o^2/c^2\right)^{3/2}$ is equal to unity, and the result becomes similar to that derived for the nonrelativistic case. For $u_o/c = \frac{1}{10}$,

$$\left(1 - \frac{u_o^2}{c^2}\right)^{3/2} = 0.999 \quad .$$

At a beam voltage of 10^4 volts,

$$\left(1 - \frac{u_o^2}{c^2}\right)^{3/2} = \left(1 - \frac{1}{25}\right)^{3/2} = 0.94 \quad .$$

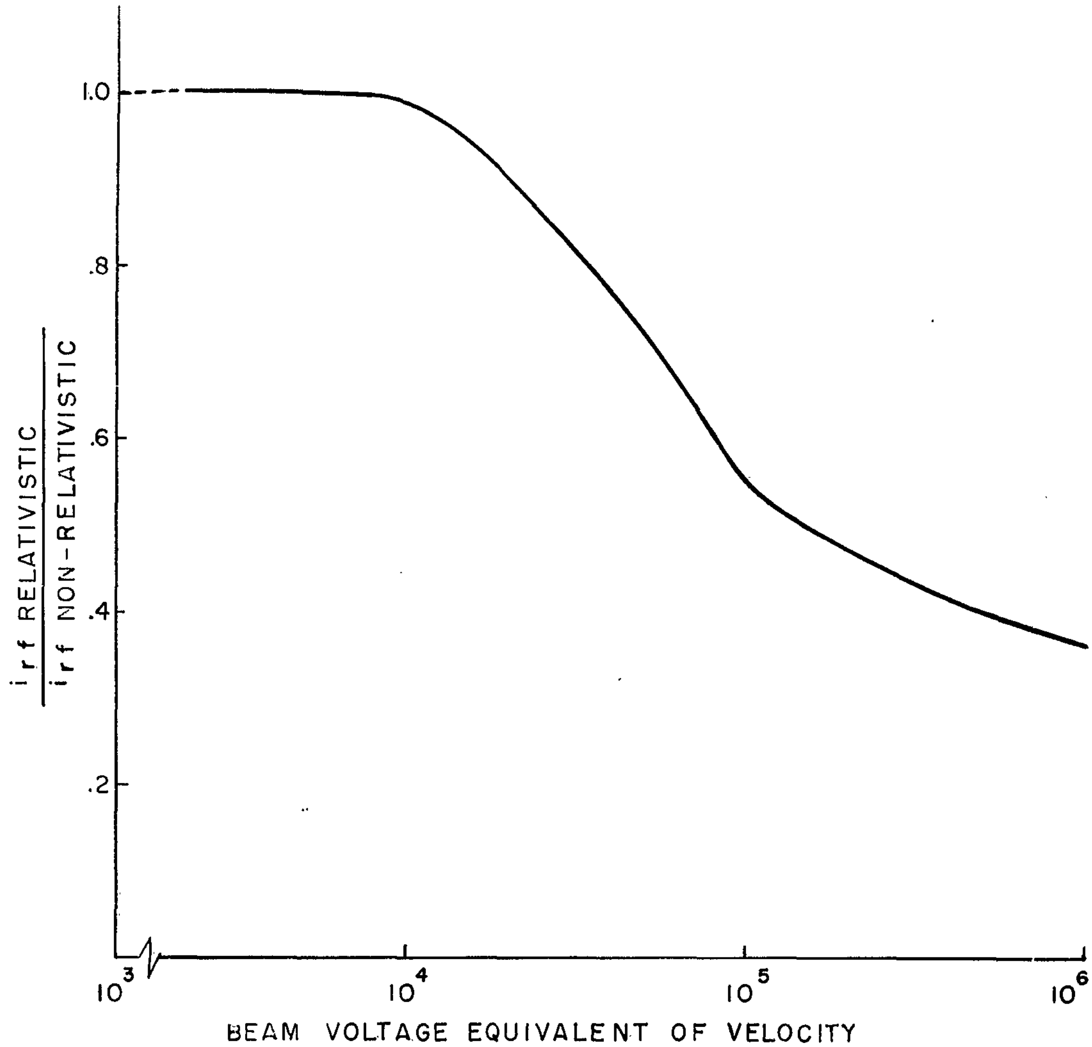


Figure 2. Graph Showing Relativistic Effects on Current Response.

III. BALLISTIC ANALYSIS

A. FIRST-ORDER BUNCHING THEORY

The problem under consideration is formulated as follows: A beam of parallel electrons which have been accelerated through a potential of V_o volts is passed through the grids of a resonator across which there appears a voltage $V_1 e^{-at^2} \sin\omega t$. The resultant electric field is assumed parallel to the electron motion. Since the velocity of an electron is proportional to the square root of the voltage through which it has been accelerated, the velocity with which an electron emerges from the first, or bunching, resonator of a two-resonator klystron will be[1]

$$V_a = u_o \sqrt{1 + \frac{\beta_1 V_1}{V_o} e^{-at^2} \sin\omega t} \;, \quad = v_o \sqrt{1 + \alpha_1 \beta_1 e^{-at^2} \sin\omega t} \quad .$$

where $u_o = \sqrt{\frac{2e}{m} V_o}$ is the d-c beam velocity, β_1 is the gap-coupling coefficient (taking into account the effect of the gap transit angle), and $\alpha_1 = V_1/V_o$. Here β_1 may not be related to the gap transit angle in the same way as for the sinusoidal case, but it obeys the general definition of the ratio of the velocity gained in the real gap with V_1 across it to the velocity gained in an infinitely narrow gap with V_1 across it and, as such, is always less than 1. The time taken by an electron to move a certain distance along the beam depends upon the point on the cycle at which it passed through the resonator gap as well as upon the magnitude of the gap voltage. If S_1 is the drift length, t_o, the time at which the electron leaves the first resonator, and t_2 the time of arrival at the catcher; then

$$t_2 = t_o + \frac{S_1}{u_o\left(1 + \alpha_1\beta_1 e^{-at_o^2} \sin\omega t_o\right)^{1/2}} \quad .$$

Now, if the modulation factor α_1 is small compared to unity; then the following approximation is reasonably valid:

$$t_2 = t_o + \frac{S_1}{u_o}\left(1 - \frac{\alpha_1\beta_1}{2} e^{-at_o^2} \sin\omega t_o\right) \quad .$$

To find the current associated with the electron bunches, one must remember that the principle of conservation of charge applies to electron bunches for an interval with corresponding departure and arrival times. The electron stream is subject to the conservation of charge, so that

$$\left|I_o dt_o\right| = \left|i_2 dt_2\right| \quad ,$$

where i_2 is the catcher current; hence

$$i_2 = \frac{I_o}{\left|\dfrac{dt_2}{dt_o}\right|}$$

$$= \frac{I_o}{1 + \dfrac{S_1}{u_o}\dfrac{\alpha_1\beta_1}{2}\left[-\left(-2\,at_o\, e^{-at_o^2}\sin\omega t_o + \omega e^{-at_o^2}\cos\omega t_o\right)\right]}$$

$$= \frac{I_o}{1 - k\left(e^{-a\theta_o^2/\omega^2}\cos\theta_o - 2\dfrac{a\theta_o}{\omega^2}e^{-a\theta_o^2/\omega^2}\sin\theta_o\right)} \quad ,$$

where the bunching parameter, $\frac{\omega S_1}{u_o} \frac{\alpha_1 \beta_1}{2} = S_1 \frac{\alpha_1 \beta_1}{2}$. As this expression is aperiodic, it cannot be represented by Fourier series, contrary to the analysis carried out by Beck[2] for the pure sine wave. Since the catcher response is desired, it has been found convenient to plot the output current versus t_2 for different bunching parameters, similar to the treatment given by Spangenburg[1] for the sinusoidal excitation. Since $i_2 = f(\theta_o) = f(-\theta_o)$ the curves are expected to be symmetrical. If the Gaussian spectrum is represented by $V e^{-at^2} \cos \omega t$, then

$$i_2 = \frac{I_o}{1 + k \left[e^{-a\theta_o^2/\omega^2} \sin \theta_o + 2 \frac{a\theta_o}{\omega^2} e^{-a\theta_o^2/\omega^2} \cos \theta_o \right]} .$$

In this case $i_2 = f(\theta_o) \neq f(-\theta_o)$. For negative values of θ_o, we have

$$i_2 = \frac{I_o}{1 - k \left[e^{-a\theta_o^2/\omega^2} \sin \theta_o + 2 \frac{a\theta_o}{\omega^2} e^{a\theta_o^2/\omega^2} \cos \theta_o \right]} .$$

The choice of a/ω^2 is governed by the following consideration. The envelope has its maximum value at $t = 0$ (since the envelope is $V e^{-at^2} \cos \omega t$) and is supposed to fall to $1/e$ of its maximum when $\omega t = 10\pi$, so that 10 r-f cycles are enclosed between the points where the amplitude is $1/e$ of the maximum value; therefore

$$\frac{a}{\omega^2} (10\pi)^2 = 1 \qquad \frac{a}{\omega^2} = \frac{1}{(10\pi)^2} .$$

Graphs have been drawn for the output current versus exit time for different values of bunching parameter for the envelopes $V_1 e^{-at^2} \sin\omega t$, and $V_1 e^{-at^2} \cos\omega t$. In the first case the infinite peaks occur for values of θ_o satisfying the transcendental equation,

$$\left(\cos\theta_o - 2\frac{a\theta_o}{2} \sin\theta_o\right) = \frac{1}{k} e^{a\theta_o^2} .$$

The values of θ_o at which infinite peaks occur are found graphically: For $k < 1$, there are no infinite peaks, as is evident from the equations also (Figure 3); for $k = 1$, one infinite peak occurs (Figure 4); for $k = 1.5$, there are 14 infinite peaks (Figure 5); for $k = 2$, there are 18 infinite peaks.*

This can be justified as follows. In the pure sine-wave case, two infinite peaks occur for $k > 1$; hence, for simplicity, we associate two infinite peaks with two peaks of the excitation signal. In the Gaussian envelope, the seventh peak occurs on either side of $t_o = 0$, when $\omega t_o = \pm 6.5\pi$. When $\omega t_o = 10\pi$, then $e^{-at_o^2} = 1/e$; therefore when $\omega t_o = 6.5\pi$, then

$$e^{-at_o^2} = \frac{1}{e^{0.425}} .$$

---

*The response for $k = 2$ has not been actually plotted, for it is not expected to be very dissimilar from the $k = 1.5$ response plot. The number of infinite peaks, however, were determined by finding the points of intersection of the curves:

$$\left(\cos\theta_o - 2\frac{a\theta_o}{\omega^2} \sin\theta_o\right) \quad \text{and} \quad \frac{1}{2} e^{a\theta_o^2} .$$

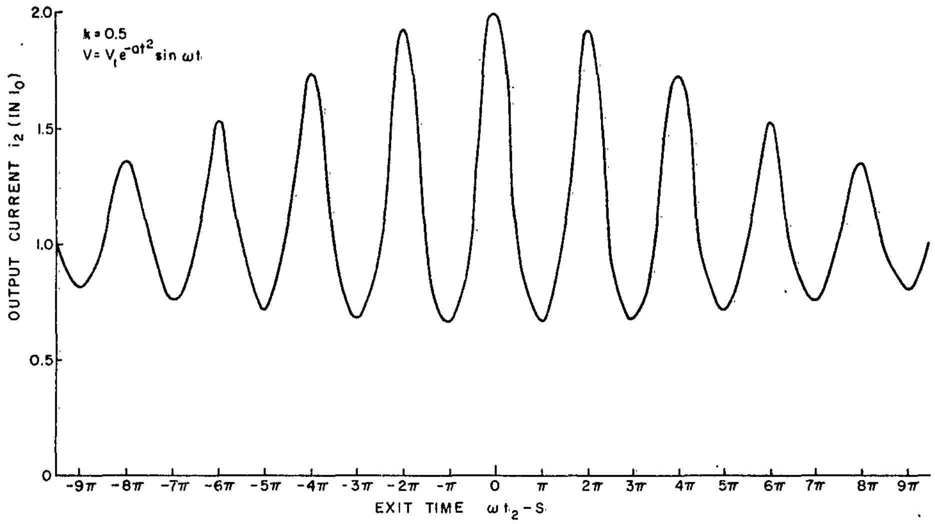


Figure 3. Output Current i_2 versus Exit Time $\omega t_2 - S$.

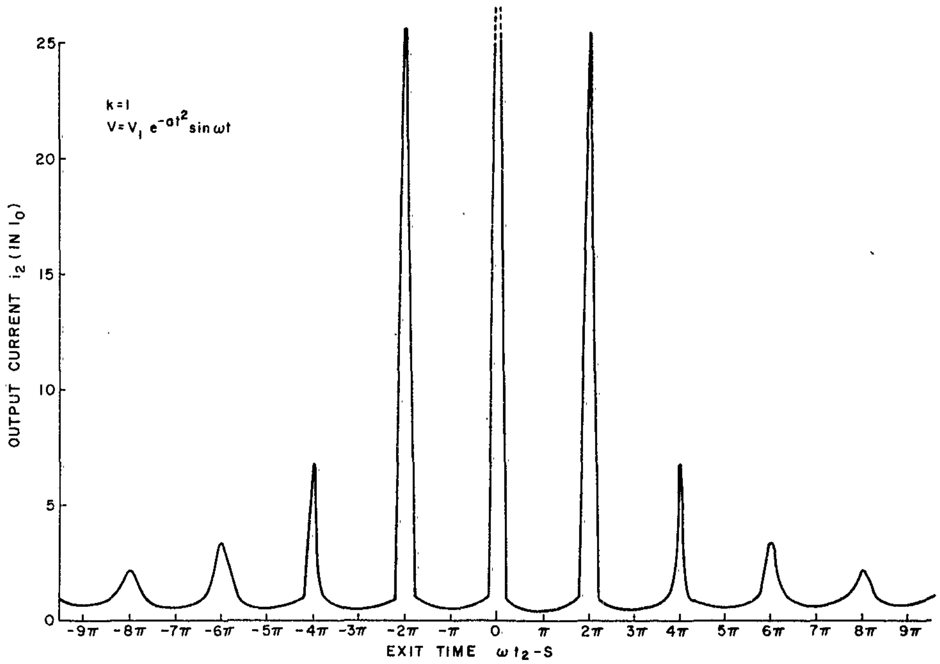


Figure 4. Output Current i_2 versus Exit Time $\omega t_2 - S$.

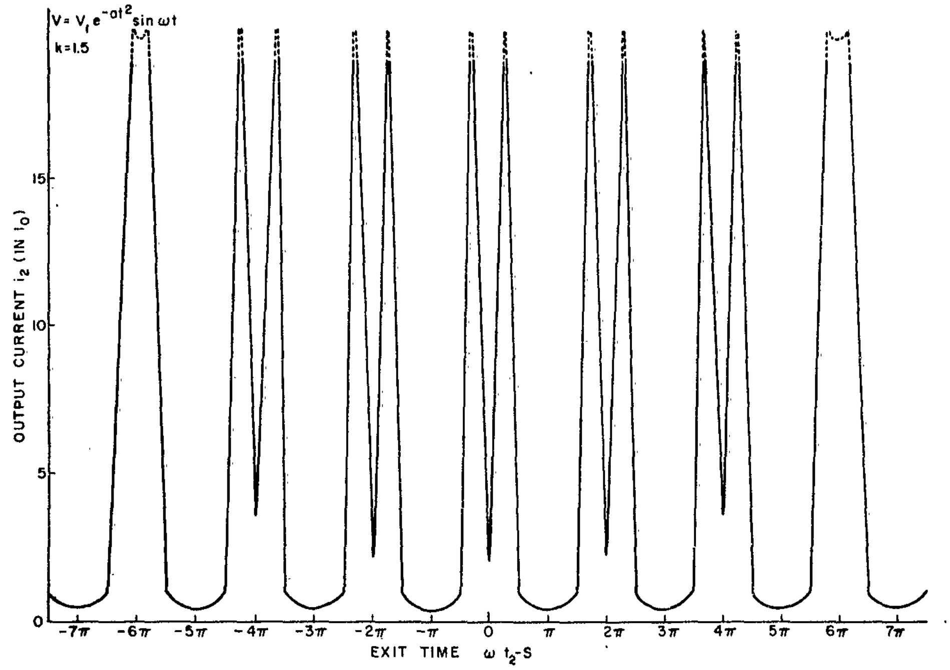


Figure 5. Output Current i_2 versus Exit Time $\omega t_2 - S$.

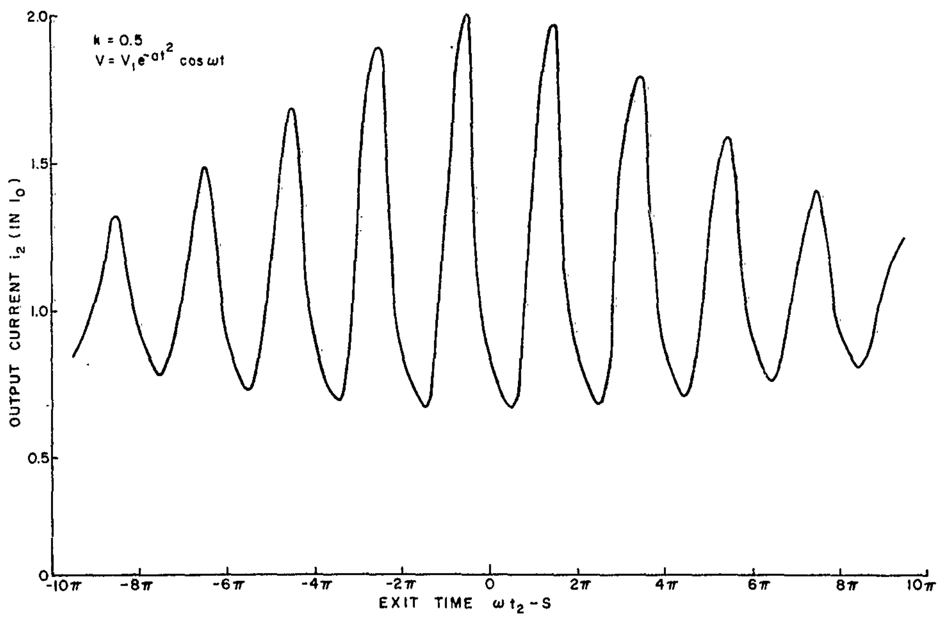


Figure 6. Output Current i_2 versus Exit Time $\omega t_2 - S$.

The value of k when $\omega t_o = 6.5\pi$ (if $k_o = 1.5$, when $t_o = 0$) is

$$k_{6.5} = \frac{1.5}{e^{0.425}} = \frac{1.5}{1.5} = 1 \quad .$$

Hence, 14 peaks are enclosed in the region for which $k > 1$.

Again, the ninth peak occurs on either side of $t_o = 0$, when $\omega t_o = 8.5\pi$, so that this region encloses 18 peaks. The value of k at $\omega t_o = 8.5\pi$ (if $k_o = 2$, at $t_o = 0$) is

$$k_{8.5} = \frac{2}{e^{0.725}} = \frac{2}{2.02} \simeq 1 \quad .$$

Thus, the occurrence of 18 infinite peaks for $k = 2$ is justified.

If the envelope is $V_1 e^{-at^2} \cos \omega t$, no infinite peaks occur for $k \leqq 1$ (Figures 6 and 7). For $k = 1.5$, there are 12 infinite peaks (Figure 8). This has also been justified by a process similar to the $V_1 e^{-at^2} \sin \omega t$ case.

B. KLYSTRON RESPONSE WITH VARIABLE GAP LENGTH

From Newton's equation of motion,

$$m \frac{d^2 z}{dt} = \frac{e V_1}{d} e^{-at^2} \cos \omega t \quad ;$$

hence

$$\frac{dz}{dt} = \int^{t} \frac{e V_1}{md} e^{-at^2} \cos \omega t \; dt + C \quad ,$$

where C is a constant. When $t = t_o$, then $dz/dt = u_o$. Thus,

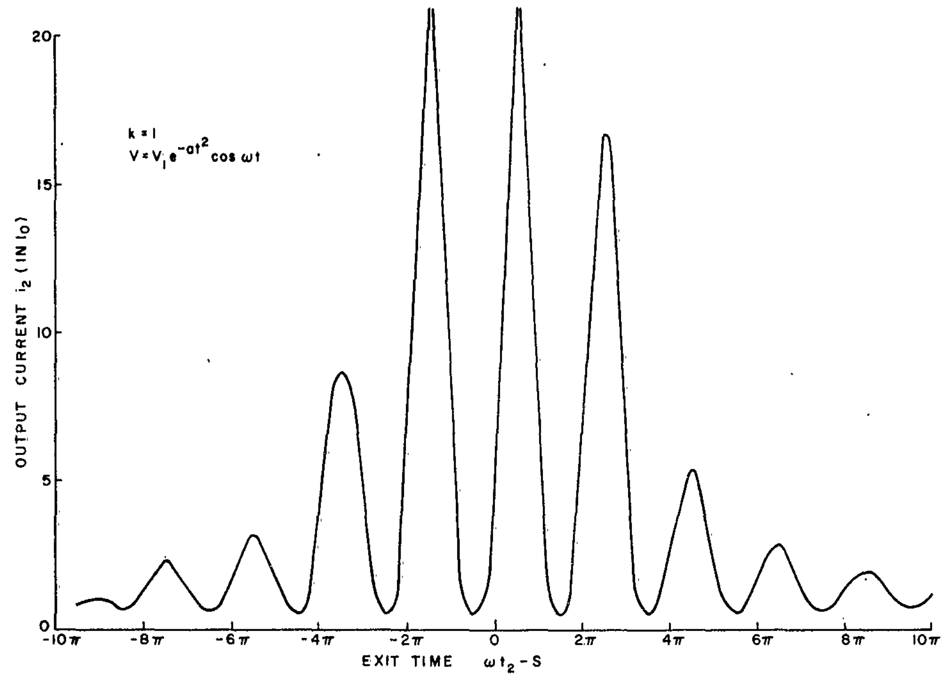


Figure 7. Output Current i_2 versus Exit Time $\omega t_2 - S$.

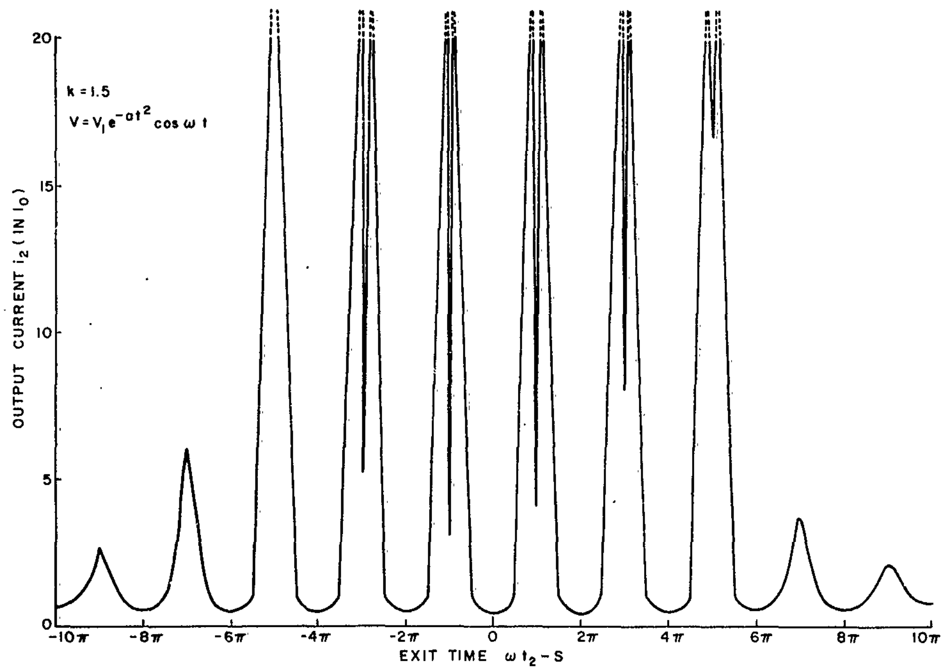


Figure 8. Output Current i_2 versus Exit Time $\omega t_2 - S$.

$$u_o' - \int^{t_o} \frac{eV_1}{md} e^{-at^2} \cos \omega t \; dt = C \quad ,$$

from which

$$\frac{dz}{dt} = u_o + \int_{t_o}^{t} \frac{eV_1}{md} e^{-at^2} \cos \omega t \; dt \quad ;$$

hence the exit velocity u_1 from the first gap at $t = t_1$ is

$$u_1 = u_o + \int_{t_o}^{t_1} \frac{eV_1}{md} e^{-at^2} \cos \omega t \; dt \quad .$$

The time of arrival at the catcher, if S_1 is the drift distance, is

$$t_2 = t_o + \frac{S_1}{u_o + \frac{eV_1}{md} \int_{t_o}^{t_1} e^{-at^2} \cos \omega t' \, dt} \quad ;$$

therefore

$$t_2 = t_o + \frac{S_1}{u_o} \left(1 + \frac{eV_1}{md\,u_o} \int_{t_o}^{t_1} e^{-at^2} \cos \omega t \; dt \right)^{-1} \quad .$$

Here an approximation will be made to permit analytical computation: $t_1 = t_o + r$, where the transit angle r is $(d/u_o) + (\delta/\omega)$.

We assume here that $\delta = 0$, so that $r = d/u_o$. Again for small modulation, the following expansion is permissible, at least for a first approximation:

$$t_2 = t_o + \frac{S_1}{u_o}\left(1 - \frac{eV_1}{mdu_o}\int_{t_o}^{t_o+r} e^{-at^2}\cos\omega t \; dt\right)$$

$$\frac{dt_2}{dt_o} = 1 + \frac{S_1}{u_o^2}\,\frac{eV_1}{md}\left[-e^{-a(t_o+r)^2}\cos\omega(t_o+r) + e^{-at_o^2}\cos\omega t_o\right] .$$

$$\frac{S_1}{u_o^2}\,\frac{eV_1}{md} = \frac{k'}{\phi_o} ,$$

where $\phi_o = \omega d/u_o$, and $k' = S_1\alpha_1/2$, hence,

$$\frac{dt_2}{dt_o} = 1 - \frac{k'}{\phi_o}\left[e^{-\frac{a}{\omega^2}(\theta_o+\phi_o)^2}\cos(\theta_o+\phi_o) - e^{-\frac{a}{\omega^2}\theta_o^2}\cos\theta_o\right] ;$$

thus

$$i_2 = \frac{I_o}{\frac{dt_2}{dt_o}} = \frac{I_o}{1 - \frac{k'}{\phi_o}\left[e^{-\frac{a}{\omega^2}(\theta_o+\phi_o)^2}\cos(\theta_o+\phi_o - e^{-\frac{a}{\omega^2}\theta_o^2}\cos\theta_o\right]} .$$

The above expression for i_2, though approximate, does give an idea of the catcher response at least for small signals.*

---

*The expression for i_2 becomes a poor approximation for large gap angles also. The graph of i_2 versus θ_o with $\phi_o = \pi/2$ has been plotted (Figure 10), taking $k = 0.5$, and 1.0; and the difference between this and the first-order bunching theory is obvious. It has been found convenient to plot i_2 versus θ_o, in this case, rather than i_2 versus θ_2, as we are only interested in an approximate estimate of the response curve.

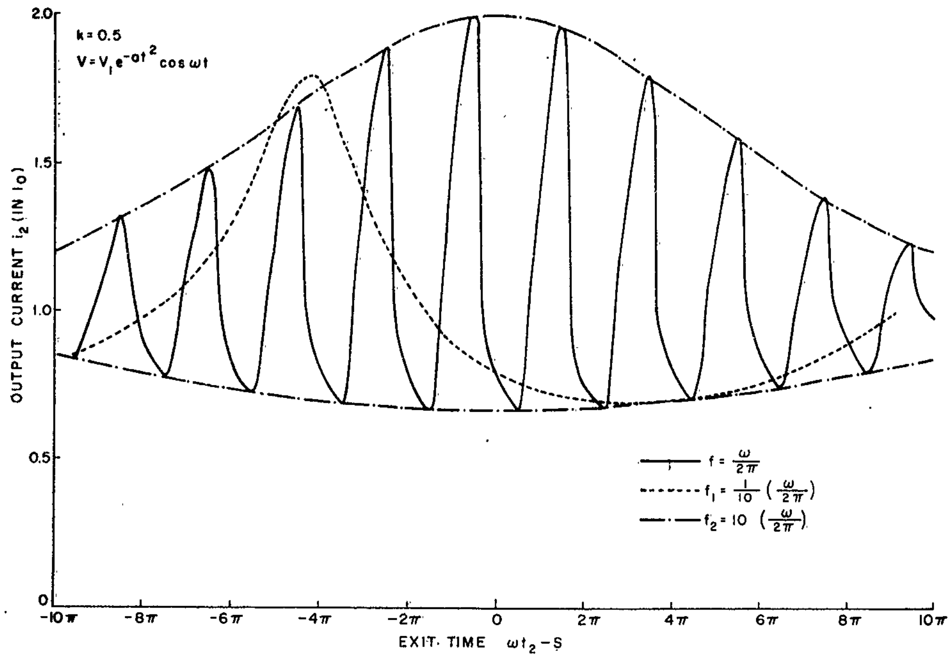


Figure 9. Output Current i_2 versus Exit Time $\omega t_2 - S$.

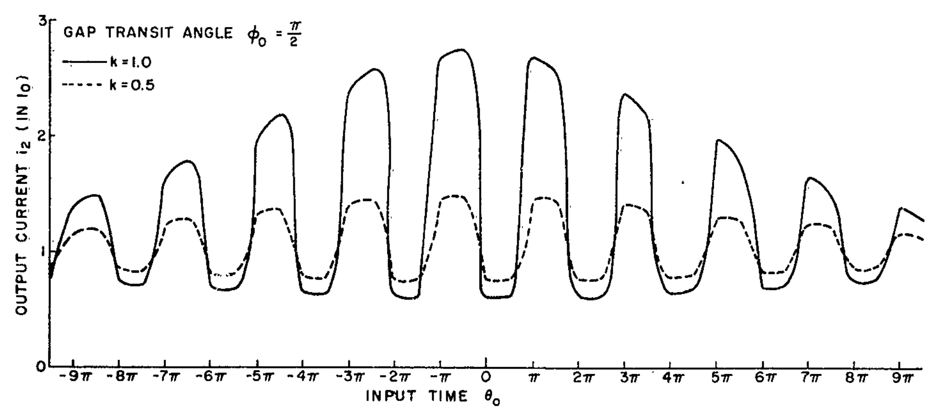


Figure 10. Output Current i_2 versus θ_0.

C. RESPONSE WITH FIXED ENVELOPE AND VARYING FREQUENCY

Using the first-order bunching theory, and taking $k = 0.5$, we have plotted the catcher current response for a fixed envelope for three different r-f frequencies (Figure 9). It has been found that the shape of the response envelope is almost independent of the change in frequency, especially at high values of $\omega/\sqrt{a}$.

IV. LARGE SIGNALS AND FINITE GAPS

The ballistic analysis for large signals with finite gaps is carried out as follows. Analysis is based on a two-cavity klystron (Figure 11).

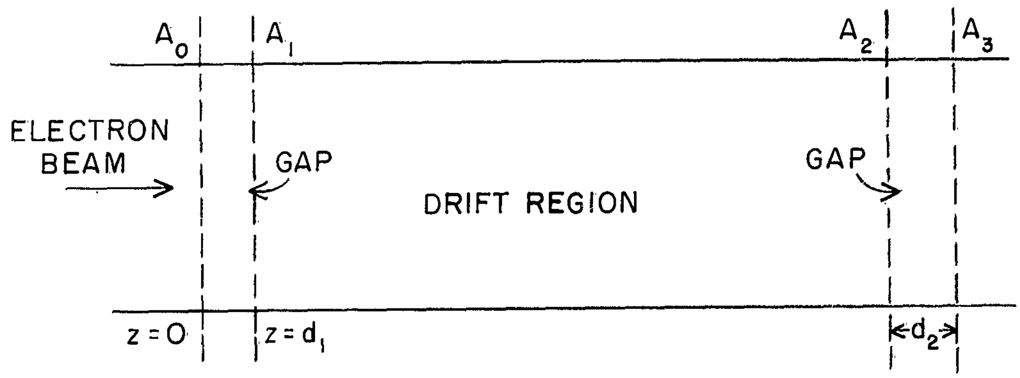


Figure 11. Schematic of Model for Velocity-modulated Tube.

In the diagram, A_o, A_1, A_2, A_3 represent the four planes under consideration. The numbers 0, 1, 2, 3 will be used to identify quantities in the respective planes.

A. FIRST GAP REGION

1. Induced R-F Current

Confining our attention first to the motion of an electron in the first gap, excited by a Gaussian pulse $V_1 e^{-at^2} \cos\omega t$, and applying Newton's second law of motion, we get

$$\frac{d^2z}{dt^2} = \frac{eV_1}{md} e^{-at^2} \cos\omega t \qquad ;$$

therefore

$$\frac{dz}{dt} = K_1 \operatorname{Re} \int e^{-at^2 + j\omega t}\, dt + C_1$$

$$= K_1\, e^{-\frac{\omega^2}{4a}} \operatorname{Re} \int e^{-a\left(t - \frac{j\omega}{2a}\right)^2} dt + C_1 \quad ,$$

where $K_1 = eV_1/md_1$ and C_1 is a constant. Now at $t = t_o$, $dz/dt = u_o$, from which C_1 is evaluated; thus

$$\frac{dz}{dt} = K_1\, e^{-\frac{\omega^2}{4a}} \operatorname{Re} \int_{t_o}^{t} e^{-a\left(t - \frac{j\omega}{2a}\right)^2} dt + u_o$$

$$= u_o + \frac{K_1\, e^{-\omega^2/4a}\sqrt{\pi}}{\sqrt{a}\; 2} \operatorname{Re}\left[\operatorname{erf}\sqrt{a}\left(t - \frac{j\omega}{2a}\right) - \operatorname{erf}\sqrt{a}\left(t_o - \frac{j\omega}{2a}\right)\right] \quad . \tag{4.1a}$$

$$\frac{dz}{dt} = u_o + \frac{K_1 e^{-\omega^2/4a}\sqrt{\pi}}{2\sqrt{a}} \left\{ \operatorname{erf}\sqrt{a}\, t - \operatorname{erf}\sqrt{a}\, t_o + \frac{2}{\sqrt{\pi}} \int_0^{\frac{\omega}{2\sqrt{a}}} \left[e^{-(at^2 - y^2)} \sin\sqrt{a}\, ty - e^{-(at_o^2 - y^2)} \sin\sqrt{a}\, ty \;\; dy \right] \right\} \quad . \tag{4.1b}$$

Integrating again, we get

$$z = u_o t + \frac{K_1}{2}\sqrt{\frac{\pi}{a}}\, e^{-\omega^2/4a} \left\{ \operatorname{erf}\sqrt{a}\, t - \operatorname{erf}\sqrt{a}\, t_o + \frac{2}{\sqrt{\pi}} \int_0^{\frac{\omega}{2\sqrt{a}}} \left[e^{-(at^2 - y^2)} \sin\sqrt{a}\, ty - e^{-(at_o^2 - y^2)} \sin\sqrt{a}\, t_o y \right] dy \right\} dt + C_2 \quad ,$$

where C_2 = constant. This constant is evaluated by using the condition that at $z = 0$, $t = t_o$; therefore

$$z = u_o(t-t_o) + \frac{K_1}{2}\sqrt{\frac{\pi}{a}}\, e^{-\omega^2/4a}\left[\int_{t_o}^{t} \operatorname{erf}\sqrt{a}\, t\, dt + \frac{2}{\sqrt{\pi}}\int_{t_o}^{t}\int_{0}^{\frac{\omega}{2\sqrt{a}}} e^{-(at^2-y^2)}\sin\sqrt{a}\, ty\, dy\, dt\right]$$

$$- \frac{K_1}{2}\sqrt{\frac{\pi}{a}}\, e^{-\omega^2/4a}\left[(t-t_o)\left\{\operatorname{erf}\sqrt{a}\, t + \frac{2}{\sqrt{\pi}}\int_{0}^{\frac{\omega}{2\sqrt{a}}} e^{-(at_o^2-y^2)}\sin\sqrt{a}\, t_o y\, dy\right\}\right] . \tag{4.2}$$

When $t = t_1$, $z = d_1$. Also the transit time is $\Gamma_1 = t_1 - t_o$; therefore

$$d_1 = u_o\Gamma_1 + \frac{K_1}{2}\sqrt{\frac{\pi}{a}}\, e^{-\omega^2/4a}\left[\int_{t_o}^{t_1} \operatorname{erf}\sqrt{a}\, t\, dt + \frac{2}{\sqrt{\pi}}\int_{t_o}^{t_1}\int_{0}^{\frac{\omega}{2\sqrt{a}}} e^{-(at^2-y^2)}\sin\sqrt{a}\, ty\, dy\, dt\right]$$

$$- \frac{K_1}{2}\sqrt{\frac{\pi}{a}}\, e^{-\omega^2/4a}\,\Gamma_1\left[\operatorname{erf}\sqrt{a}\, t_o + \frac{2}{\sqrt{\pi}}\int_{0}^{\frac{\omega}{2\sqrt{a}}} e^{-(at_o^2-y^2)}\sin\sqrt{a}\, t_o y\, dy\, dt\right] . \tag{4.3}$$

Suppose $\Gamma_1 = \frac{d_1}{u_o} + \frac{\delta_1}{\omega}$, where δ_1 is the correction factor; then $\omega d_1/u_o =$ $= \phi_1$ is the d-c transit angle. From Equation (4.3),

$$\phi_1 = \phi_1 + \delta_1 + \frac{K_1\omega}{2u_o}\sqrt{\frac{\pi}{a}}\, e^{-\omega^2/4a}\left[\int_{t_o}^{t_1} \operatorname{erf}\sqrt{a}\, t\, dt + \frac{2}{\sqrt{\pi}}\int_{t_o}^{t_1}\int_{0}^{\frac{\omega}{2\sqrt{a}}} e^{-(at^2-y^2)}\sin\sqrt{a}\, ty\, dy\, dt\right]$$

$$- \frac{K_1\sqrt{\pi}}{2\sqrt{a}} \frac{e^{-\omega^2/4a}}{u_o} (\phi_1 + \delta_1) \left[\mathrm{erf}\sqrt{a}\, t_o + \frac{2}{\sqrt{\pi}} \int_0^{\frac{\omega}{2\sqrt{a}}} e^{-(at_o^2 - y^2)} \sin\sqrt{a}\, t_o y \, dy \right] .$$

Writing t_1 as t and replacing t_o by $t_1 - \Gamma_1 = t_1 - \frac{d_1}{u_o} - \frac{\delta_1}{\omega}$, we get

$$\delta_1 = \Bigg(\frac{K_1\omega}{2u_o} \sqrt{\frac{\pi}{a}}\, e^{-\omega^2/4a} \left[\int_{t - \frac{d_1}{u_o} - \frac{\delta_1}{\omega}}^{t} \mathrm{erf}\sqrt{a}\, t\, dt + \frac{2}{\sqrt{\pi}} \int_{t - \frac{d_1}{u_o} - \frac{\delta_1}{\omega}}^{t} \int_0^{\frac{\omega}{2\sqrt{a}}} e^{-(at^2 - y^2)} \sin\sqrt{a}\, ty\, dy \right]$$

$$- \frac{K_1}{2} \sqrt{\frac{\pi}{a}}\, e^{-\omega^2/4a} \frac{\phi_1}{u_o} \Bigg\{ \mathrm{erf}\sqrt{a} \left(t - \frac{d_1}{u_o} - \frac{\delta_1}{\omega}\right)$$

$$+ \frac{2}{\sqrt{\pi}} \int_0^{\frac{\omega}{2\sqrt{a}}} e^{-\left[a\left(t_1 - \frac{d_1}{u_o} - \frac{\delta_1}{\omega}\right)^2 - y^2\right]} \sin\sqrt{a}\left(t_1 - \frac{d_1}{u_o} - \frac{\delta_1}{\omega}\right) dy \Bigg\} \Bigg) .$$

$$\Bigg(1 - \frac{K_1 e^{-\omega^2/4a}}{2u_o} \sqrt{\frac{\pi}{a}} \Bigg\{ \mathrm{erf}\sqrt{a}\left(t - \frac{d_1}{u_o} - \frac{\delta_1}{\omega}\right)$$

$$+ \frac{2}{\sqrt{\pi}} \int_0^{\frac{\omega}{2\sqrt{a}}} e^{-\left[a\left(t - \frac{d_1}{u_o} - \frac{\delta_1}{\omega}\right)^2 - y^2\right]} \sin\sqrt{a}\left(t - \frac{d_1}{\omega} - \frac{\delta_1}{\omega}\right) dy \Bigg\} \Bigg)^{-1} . \quad (4.4)$$

In Equation (4.4), δ_1 is implicit, and approximation must be made to obtain an explicit expression for it.

We shall assume that δ_1 is small so that

$$\operatorname{Re}\operatorname{erf}\left[\sqrt{a}\left(t-\frac{d_1}{u_o}-\frac{\delta_1}{\omega}\right)-\frac{j\omega}{2\sqrt{a}}\right] = \operatorname{Re}\left\{\left[\sqrt{a}\left(t-\frac{d_1}{u_o}\right)-\frac{j\omega}{2\sqrt{a}}\right]\right.$$

$$\left.-\frac{\sqrt{a}\,\delta_1}{\omega}\operatorname{erf}^1\left[\sqrt{a}\left(t-\frac{d_1}{u_o}\right)-\frac{j\omega}{2\sqrt{a}}\right]\right\} ,$$

where

$$\operatorname{erf}^1\left[\sqrt{a}\left(t-\frac{d_1}{u_o}\right)-\frac{j\omega}{2\sqrt{a}}\right] = e^{-\left[\sqrt{a}\left(t-\frac{d_1}{u_o}\right)-\frac{j\omega}{2\sqrt{a}}\right]^2} .$$

Now as Γ_1 is small compared to t, the following approximation is justified:

$$\int_{t-\frac{d_1}{u_o}-\frac{\delta_1}{\omega}}^{t}\operatorname{erf}\sqrt{a}\,t\,dt = \left(\frac{d_1}{u_o}+\frac{\delta_1}{\omega}\right)\operatorname{erf}\sqrt{a}\,t$$

$$\int_{t-\frac{d_1}{u_o}-\frac{\delta_1}{\omega}}^{t}\int_{o}^{\frac{\omega}{2\sqrt{a}}} e^{-(at^2-y^2)}\sin\sqrt{a}\,ty\,dy\,dt = \left(\frac{d_1}{u_o}+\frac{\delta_1}{\omega}\right)\int_{o}^{\frac{\omega}{2\sqrt{a}}} e^{-(at^2-y^2)}\sin\sqrt{a}\,ty\,dy .$$

Substituting these in Equation (4.4), we have

$$\delta_1 = -\left(\frac{K_1\omega}{2u_o}\sqrt{\pi/a}\,e^{-\frac{\omega^2}{4a}}\left(\frac{d_1}{u_o}+\frac{\delta_1}{\omega}\right)\operatorname{Re}\left[\operatorname{erf}\left(\sqrt{a}\,t-\frac{j\omega}{2\sqrt{a}}\right)\right] -\right.$$

$$- \frac{K_1}{2}\sqrt{\frac{\pi}{a}}\, e^{-\frac{\omega^2}{4a}} \frac{\phi_1}{u_o} \, \mathrm{Re}\left\{\mathrm{erf}\left[\sqrt{a}\left(t - \frac{d_1}{u_o}\right) - \frac{j\omega}{2\sqrt{a}}\right] - \sqrt{a}\,\frac{\delta_1}{\omega}\left[\mathrm{erf}^1 \sqrt{a}\left(t - \frac{d_1}{u_o}\right) - \frac{j\omega}{2\sqrt{a}}\right]\right\}\bigg)$$

$$\left(1 - \frac{K}{2}\sqrt{\frac{\pi}{a}}\, \frac{e^{-\frac{\omega^2}{4a}}}{u_o} \, \mathrm{Re}\left\{\mathrm{erf}\left[\sqrt{a}\left(t - \frac{d_1}{u_o}\right) - \frac{j\omega}{2\sqrt{a}}\right] - \sqrt{a}\,\frac{\delta_1}{\omega}\, \mathrm{erf}^1\left[\sqrt{a}\left(t - \frac{d_1}{u_o}\right) - \frac{j\omega}{2\sqrt{a}}\right]\right\}\right)^{-1} .$$

This equation obviously gives a quadratic in δ_1, which might be complicated to solve. For a simpler solution, we will neglect the δ_1^2 term; therefore

$$\delta_1 = -\left[\frac{K_1}{2}\sqrt{\frac{\pi}{a}}\, e^{-\frac{\omega^2}{4a}} \frac{\phi_1}{u_o}\left(\mathrm{Re}\left\{\mathrm{erf}\left(\sqrt{a}\, t - \frac{j\omega}{2\sqrt{a}}\right) - \mathrm{erf}\left[\sqrt{a}\left(t - \frac{d_1}{u_o}\right) - \frac{j\omega}{2\sqrt{a}}\right]\right\}\right)\right]$$

$$\left[1 + \frac{K_1}{2}\sqrt{\frac{\pi}{a}}\, \frac{e^{-\frac{\omega^2}{4a}}}{u_o}\left(\mathrm{Re}\left\{\mathrm{erf}\left(\sqrt{a}\, t - \frac{j\omega}{2\sqrt{a}}\right)\right.\right.\right.$$

$$\left.\left.\left. - \mathrm{erf}\left[\sqrt{a}\left(t - \frac{d_1}{u_o}\right) - \frac{j\omega}{2\sqrt{a}}\right] + \left[\frac{d_1\sqrt{a}}{u_o}\, \mathrm{erf}^1 \sqrt{a}\left(t - \frac{d_1}{u_o}\right) - \frac{j\omega}{2\sqrt{a}}\right]\right\}\right)\right]^{-1} . \tag{4.5}$$

Thus Equation (4.5) gives an explicit expression for δ_1. Denoting

$$F(t) = \frac{K_1}{2}\sqrt{\frac{\pi}{a}}\, e^{-\frac{\omega^2}{4a}} \frac{\phi_1}{u_o}\left(\mathrm{Re}\left\{\mathrm{erf}\left(\sqrt{a}\, t - \frac{j\omega}{2\sqrt{a}}\right) - \mathrm{erf}\left[\sqrt{a}\left(t - \frac{d_1}{u_o}\right) - \frac{j\omega}{2\sqrt{a}}\right]\right\}\right) ,$$

$$G(t) = \frac{K_1}{2}\sqrt{\frac{\pi}{a}}\, e^{-\frac{\omega^2}{4a}} \frac{d_1\sqrt{a}}{u_o^2}\, \mathrm{Re}\, \mathrm{erf}^1\left[\sqrt{a}\left(t - \frac{d_1}{u_o}\right) - \frac{j\omega}{2\sqrt{a}}\right] ,$$

gives

$$\delta_1 = - \frac{F(t)}{1 + \frac{1}{\phi_1} F(t) + G(t)} . \tag{4.6}$$

According to Ramo's theorem, the current at time t resulting from the charge entering the gap in the interval dt_o between time t_o and $t_o + dt_o$ is

$$di_1 = \frac{I_o}{d_1} u \, dt_o \quad ,$$

where u = velocity at time t. The total current induced is

$$i_1 = I_o/d_1 \int_{t-\Gamma_1}^{t} u \, dt_o \quad .$$

Substituting Equation (4.1b) in the preceding integral, we have

$$i_1 = \frac{I_o}{d_1} \int_{t-\Gamma_1}^{t} \left(u_o + \frac{K_1}{2} \sqrt{\frac{\pi}{a}} \, e^{-\frac{\omega^2}{4a}} \left[\operatorname{erf}\sqrt{a}\,t - \operatorname{erf}\sqrt{a}\,t_o + \frac{2}{\sqrt{\pi}} \int_0^{\frac{\omega}{2\sqrt{a}}} e^{-(at^2 - y^2)} \sin\sqrt{a}\,ty \right.\right.$$

$$\left.\left. - e^{-(at_o^2 - y^2)} \sin\sqrt{a}\,t_o y \, dy \right] \right)$$

$$= \frac{I_o}{d_1} \left\{ u_o \Gamma + \frac{K_1}{2} \sqrt{\frac{\pi}{a}} \, e^{-\frac{\omega^2}{4a}} \, \Gamma \left[\operatorname{erf}\sqrt{a}\,t + \frac{2}{\sqrt{\pi}} \int_0^{\frac{\omega}{2\sqrt{a}}} e^{-(at^2 - y^2)} \sin\sqrt{a}\,ty \quad dy \right] \right.$$

$$\left. - \frac{K_1}{2} \sqrt{\frac{\pi}{a}} \, e^{-\frac{\omega^2}{4a}} \left[\int_{t-\Gamma_1}^{t} \operatorname{erf}\sqrt{a}\,t_o dt_o + \frac{2}{\sqrt{\pi}} \int_{t-\Gamma_1}^{t} \int e^{-(at_o^2 - y^2)} \sin\sqrt{a}\,t_o y \quad dy \, dt_o \right] \right\} ;$$

therefore

$$i_1 = I_o + \frac{I_o}{d_1} \left\{ u_o \frac{\delta_1}{\omega} + \frac{K_1\sqrt{\pi}}{2\sqrt{a}} \left(\frac{d_1}{u_o} + \frac{\delta_1}{\omega} \right) e^{-\frac{\omega^2}{4a}} \left[\operatorname{erf}\sqrt{a}\,t + \frac{2}{\sqrt{\pi}} \int_0^{\frac{\omega}{2\sqrt{a}}} e^{-(at^2 - y^2)} \sin\sqrt{a}\,ty \; dy \right] \right.$$

$$- \frac{K_1}{2} \sqrt{\frac{\pi}{a}}\, e^{-\frac{\omega^2}{4a}} \left[\int_{t-\Gamma_1}^{t} \operatorname{erf}\sqrt{a}\, t_o \, dt + \frac{2}{\sqrt{\pi}} \int_{t-\Gamma_1}^{t} \int e^{-(at_o^2 - y^2)} \sin\sqrt{a}\, t_o y \, dy \, dt_o \right] \Bigg\} \quad . \tag{4.7}$$

The above expression is considerably simplified if the following approximations are accepted as valid:

$$\int_{t-\Gamma_1}^{t} \operatorname{erf}\sqrt{a}\, t_o \, dt_o = \Gamma_1 \operatorname{erf}\sqrt{a}\, t \quad ,$$

$$\frac{2}{\sqrt{\pi}} \int_{t-\Gamma_1}^{t} \int_0^{\frac{\omega}{2\sqrt{a}}} e^{-(at_o^2 - y^2)} \sin\sqrt{a}\, t_o y \, dy dt_o = \frac{2}{\sqrt{\pi}}\, \Gamma_1 \int_0^{\frac{\omega}{2\sqrt{a}}} e^{-(at^2 - y^2)} \sin\sqrt{a}\, ty \, dy \quad .$$

This is very nearly true, especially for small gaps; then, Equation (4.7) reduces to

$$i_1 = I_o \left(1 + \frac{\delta_1}{\phi_1} \right)$$

$$= I_o \left\{ 1 + \frac{[-F(t)]}{1 + \frac{F(t)}{\phi_1} + G(t)} \quad \frac{1}{\phi_1} \right\} \quad . \tag{4.8}$$

It is interesting to note in this case that the time-dependent component of current i_1 is directly proportional to the correction factor δ_1.

A more accurate simplification of Equation (4.7) than that given by Equation (4.8) is obtained if

$$\int_{t-\Gamma_1}^{t} \operatorname{erf}\sqrt{a}\, t_o \, dt = \Gamma_1 \operatorname{erf}\sqrt{a} \left(t - \frac{\Gamma_1}{2} \right)$$

$$\simeq \Gamma_1 \operatorname{erf} \sqrt{a} \left(t - \frac{d_1}{2u_o} - \frac{\delta_1}{2\omega} \right) .$$

Putting this in Equation (4.7), we have

$$i_1 = I_o + \frac{I_o}{d_1} \left[\frac{u_o \delta_1}{\omega} + \frac{K_1}{2} \sqrt{\frac{\pi}{a}}\, e^{-\frac{\omega^2}{4a}} \left(\frac{d_1}{u_o} + \frac{\delta_1}{\omega} \right) \left(\mathrm{Re} \left\{ \operatorname{erf} \left(\sqrt{a}\, t - \frac{j\omega}{2\sqrt{a}} \right) - \operatorname{erf} \left[\sqrt{a} \left(t - \frac{d_1}{2u_o} - \frac{\delta_1}{2\omega} \right) - \frac{j\omega}{2\sqrt{a}} \right] \right\} \right) \right] .$$

After expansion in a Taylor series, we get

$$i_1 = I_o + \frac{I_o}{d_1} \left[\frac{u_o \delta_1}{\omega} + \mathrm{Re}\, \frac{K_1}{2} \frac{\sqrt{\pi}}{\sqrt{\pi}}\, e^{-\frac{\omega^2}{4a}} \left(\frac{d_1}{u_o} + \frac{\delta_1}{\omega_1} \right)^2 e^{-\left(\sqrt{a}\, t - \frac{j\omega}{2\sqrt{a}} \right)^2} \right]$$

$$= I_o + \frac{I_o}{d_1} \left[u_o \frac{\delta_1}{\omega} + \frac{K_1}{2} \frac{\sqrt{\pi}}{\sqrt{\pi}} \left(\frac{d_1}{u_o} + \frac{\delta_1}{\omega} \right)^2 e^{-at^2} \cos \omega t \right] .$$

If the δ_1^2 term is neglected,

$$i_1 = I_o + \frac{I_o}{d_1} \left(\frac{\delta_1}{\omega} u_o + \frac{V_1}{2V_o} \frac{u_o}{\omega} \delta_1 e^{-at^2} \cos \omega t + \frac{V_1}{4V_o} d_1 e^{-at^2} \cos \omega t \right) , \tag{4.9}$$

where V_o = d-c beam voltage $= \frac{m}{2e} u_o^2$.

2. Velocity Modulation

From Equation (4.1a), we have

$$v_1 = u_o + \frac{K_1}{2} \sqrt{\frac{\pi}{a}}\, e^{-\frac{\omega^2}{4a}}\, \mathrm{Re} \left[\operatorname{erf} \sqrt{a} \left(t_1 - \frac{j\omega}{2a} \right) - \operatorname{erf} \sqrt{a} \left(t_o - \frac{j\omega}{2a} \right) \right] .$$

Making suitable approximations to simplify solution gives

$$\frac{2}{\sqrt{\pi}} \int_{\sqrt{a}\, t_o - \frac{j\omega}{2\sqrt{a}}}^{\sqrt{a}\, t_1 - \frac{j\omega}{2\sqrt{a}}} e^{-t^2}\, dt = \frac{2\sqrt{a}}{\sqrt{\pi}} \left(\frac{d_1}{u_o} + \frac{\delta_1}{\omega}\right) e^{-\left(\sqrt{a}\, t_1 - \frac{j\omega}{2\sqrt{a}}\right)^2} \quad ;$$

from which

$$\frac{v_1}{u_o} = 1 + \frac{K_1}{2} \frac{\sqrt{\pi}}{u_o} \left(\frac{d_1}{u_o} + \frac{\delta_1}{\omega}\right) e^{-at_1^2} \cos\, \omega t_1 \quad . \tag{4.10}$$

This is justified if the gaps are very short and the transit time small. For larger gaps, a better approximation would be

$$\int_{\sqrt{a}\, t_o - \frac{j\omega}{2\sqrt{a}}}^{\sqrt{a}\, t_1 - \frac{j\omega}{2\sqrt{a}}} e^{-t^2}\, dt = \sqrt{a} \left(\frac{d_1}{u_o} + \frac{\delta_1}{\omega}\right) e^{-\left[\sqrt{a}\left(t_1 - \frac{d}{2u_o}\right) - \frac{j\omega}{2\sqrt{a}}\right]} \quad ;$$

then

$$\frac{v_1}{u_o} = 1 + \frac{K_1}{2u_o}\left(\frac{d_1}{u_o} + \frac{\delta_1}{\omega}\right) \sqrt{\pi}\; e^{-a\left(t_1 - \frac{d}{2u_o}\right)^2} \cos\left(\omega t_1 - \frac{\phi_1}{2}\right) \quad . \tag{4.11}$$

The neglect of δ_1 in the exponential is reasonable when the resulting simplification in computation is taken into consideration. Note that when $\delta_1 = 0$, Equation (4.10) reduces to

$$\frac{v_1}{u_o} = 1 + \frac{V_1}{2V_o} e^{-at_1^2} \cos \omega t_1 \quad ,$$

an expression similar to the velocity modulation in the sinusoidal excitation case.

B. DRIFT SPACE REGION

The electron beam enters the drift space with both velocity and current modulation, as shown above, and it drifts in a field-free space resulting in further increase in the harmonic content of the beam current.

Neglecting space-charge debunching, we have $t_2 = t_1 + (S_1/v_1)$, where S_1 is the drift length; therefore

$$t_2 = t_1 + \frac{S_1}{u_o \left\{1 + \frac{K_1}{2} e^{-\frac{\omega^2}{4a}} \sqrt{\frac{\pi}{a}} \left[\mathrm{erf}\left(\sqrt{a}\, t_1 - \frac{j\omega}{2\sqrt{a}}\right) - \mathrm{erf}\left(\sqrt{a}\, t_o - \frac{j\omega}{2\sqrt{a}}\right)\right]\right\}} \quad .$$

Using the simplified expression in Equation (4.10) gives

$$t_2 = t_1 + \frac{S_1}{u_o \left[1 + \frac{K_1}{2u_o}\left(\frac{d_1}{u_o} + \frac{\delta_1}{\omega}\right) \sqrt{\pi}\, e^{-at_1^2} \cos \omega t_1\right]} \quad . \tag{4.12}$$

Using Equation (4.11) results in a slight modification:

$$t_2 = t_1 + \frac{S_1}{u_o \left[1 + \frac{\sqrt{\pi}}{2} \frac{K_1}{u_o}\left(\frac{d_1}{u_o} + \frac{\delta_1}{\omega}\right) e^{-a\left(t_1 - \frac{d_1}{u_o}\right)^2} \cos\left(\omega t_1 - \frac{\phi_1}{2}\right)\right]} \quad . \tag{4.13}$$

This expression for small modulation becomes

$$t_2 = t_1 + \frac{S_1}{u_o}\left[1 - \frac{K_1}{u_o}\left(\frac{d_1}{u_o} + \frac{\delta_1}{\omega}\right) e^{-a\left(t_1 - \frac{d_1}{u_o}\right)^2} \cos\left(\omega t_1 - \frac{\phi_1}{2}\right)\right] .$$

It is difficult to differentiate this function because of the presence of δ_1. It is better to keep the integral in the expression for dz/dt rather than divided into error functions; thus

$$t_2 = t_1 + \frac{S_1}{u_o\left[1 + \frac{K_1}{u_o} e^{-\frac{\omega^2}{4a}} \operatorname{Re}\int_{t_o}^{t_1} e^{-a\left(t - \frac{j\omega}{2a}\right)^2} dt\right]}$$

$$\simeq t_1 + \frac{S_1}{u_o}\left[1 - \frac{K_1}{u_o} e^{-\frac{\omega^2}{4a}} \operatorname{Re}\int_{t_o}^{t_1} e^{-a\left(t - \frac{j\omega}{2a}\right)^2} dt\right] ,$$

$$\frac{dt_2}{dt_1} = 1 - \frac{S_1 K_1}{u_o^2} \operatorname{Re}\left\{e^{-a\left(t_1 - \frac{j\omega}{2a}\right)^2} - e^{-\left[\sqrt{a}\,(t_1 - \Gamma_1) - \frac{j\omega}{2\sqrt{a}}\right]^2}\right\} e^{-\frac{\omega^2}{4a}}$$

$$\simeq 1 - \frac{S_1}{u_o^2} K_1 \left[e^{-at_1^2} \cos \omega t_1 - e^{-a\left(t_1 - \frac{d_1}{u_o}\right)^2} \cos(\omega t_1 - \phi_1)\right] ,$$

assuming that

$$e^{-a\left(t_1 - \frac{d_1}{u_o} - \frac{\delta_1}{\omega}\right)^2} \cos(\omega t_1 - \phi_1 - \delta_1) = e^{-a\left(t_1 - \frac{d_1}{u_o}\right)^2} \cos(\omega t_1 - \phi_1) , \tag{4.14}$$

therefore

$$i_2 = \frac{i_1}{\frac{dt_2}{dt_1}} = \frac{i_1}{1 - \frac{S_1}{2d}\frac{V_1}{V_o}\left[e^{-at_1^2}\cos \omega t_1 - e^{-a\left(t_1 - \frac{d_1}{u_o}\right)^2}\cos(\omega t_1 - \phi_1)\right]} .$$

But by Equation (4.7), i_1 is related to I_o; therefore theoretically it is possible to express

$$i_2 = I_o \ f(t_1) \quad , \tag{4.15}$$

where $f(t_1)$ denotes a function in t_1. Again as t_2 is related to t_1, as shown above, it is possible to express i_2 as a function of t_2, i.e.,

$$i_2 = I_o \ g(t_2) \quad . \tag{4.16}$$

Although Equation (3.16) is expected to be very complicated, and rigid mathematical analysis seems highly improbable in practice, a theoretical formulation is not ruled out. It is desirable to expand i_2 in a series in such a way that the various frequency components become distinguishable, but the obvious aperiodicity of i_2 rules out the possibility of expanding in a Fourier series, as in the sinusoidal case analyzed by Beck.

C. SECOND GAP REGION

At this stage, attention will be directed to the motion of electrons in the second gap. The beam induces a voltage on the grids, and because the voltage produces a change in beam current and velocity, we can consider this behavior as a reciprocal relationship, so that knowing the effect of the

voltage on the beam is equivalent to knowing the effect of the changing beam upon the induced voltage. With this in mind, let us assume the voltage induced in the second gap to have a spectrum given by the equation $V_2 e^{-b(t+\rho)^2} \cos\omega(t+\rho)$, where ρ is a constant introduced to take into account the possible change of phase with respect to the original envelope $V_1 e^{-at^2} \cos\omega t$. Although this assumption might differ from the actual physical conditions, it is acceptable as a first approximation. The equation of motion in the gap can therefore be written as

$$\frac{d^2z}{dt^2} = K_2\, e^{-b(t+\rho)^2} \cos\omega(t+\rho) \quad ;$$

therefore

$$\frac{dz}{dt} = K_2 \int e^{-b(t+\rho)^2} \cos\omega(t+\rho)\, dt + C_3 \quad ,$$

where C_3 = constant. Now at $t = t_2$,

$$\frac{dz}{dt} = u_o + \frac{K_1}{2}\sqrt{\frac{\pi}{a}}\, e^{-\frac{\omega^2}{4a}} \operatorname{Re}\left[\operatorname{erf}\left(\sqrt{a}\, t_1 - \frac{j\omega}{2\sqrt{a}}\right) - \operatorname{erf}\left(\sqrt{a}\, t_o - \frac{j\omega}{2\sqrt{a}}\right)\right] \quad .$$

Using this condition to calculate C_3, we have

$$\frac{dz}{dt} = K_2 \int_{t_2}^{t} e^{-b(t+\rho)^2} \cos\omega(t+\rho)\, dt + V_2 \quad , \tag{4.17}$$

where

$$V_2 = u_o + \frac{K_1}{2}\sqrt{\frac{\pi}{a}}\, e^{-\frac{\omega^2}{4a}} \operatorname{Re}\left[\operatorname{erf}\left(\sqrt{a}\, t_1 - \frac{j\omega}{2a}\right) - \operatorname{erf}\left(\sqrt{a}\, t_o - \frac{j\omega}{2\sqrt{a}}\right)\right] \quad .$$

Integrating again, we have

$$z = K_2 \int_{t_2}^{t} e^{-b(t+\rho)^2} \cos \omega(t+\rho)\, dt + V_2 t + C_4 \quad , \qquad (4.18)$$

where C_4 = constant. At $t = t_2$, $z = d_1 + S_1$; therefore

$$z = K_2 \int_{t_2}^{t} \int_{t_2}^{t} e^{-b(t+\rho)^2} \cos \omega(t+\rho)\, dt\, dt + V_2(t - t_2) + d_1 + S_1 \quad .$$

Now at $z = d_2 + S_1 + d_1$, we have $t = t_3$. Also $t_3 - t_2 = \Gamma_2$ is the transit time in the second gap; therefore

$$d_2 = K_2 \int_{t-\Gamma_2}^{t_3} \frac{e^{-\frac{\omega^2}{4b}}}{2} \sqrt{\frac{\pi}{b}}\, \mathrm{Re} \left\{ \mathrm{erf} \left[\sqrt{b}\,(t+\rho) - \frac{j\omega}{2\sqrt{b}} \right] - \mathrm{erf} \left[\sqrt{b}\,(t_2+\rho) - \frac{j\omega}{2\sqrt{b}} \right] \right\} dt + V_2 \Gamma_2 \quad .$$

Suppose

$$\Gamma_2 = \frac{d_2}{u_2} + \frac{\delta_2}{\omega} \quad , \quad \text{and} \quad \frac{\omega d_2}{V_2} = \phi_2 \quad ,$$

where δ_2 is a correction factor for second gap. Then, by a process similar to that used for the first gap, and with similar approximations, we have

$$\delta_2 = - \left[\frac{K_2 e^{-\frac{\omega^2}{4b}}}{2 V_2} \sqrt{\frac{\pi}{b}}\, \phi_2\, \mathrm{Re} \left\{ \mathrm{erf} \left[\sqrt{b}\,(t+\rho) - \frac{j\omega}{2\sqrt{b}} \right] - \mathrm{erf} \sqrt{b}\,(t_2+\rho) - \frac{j\omega}{2\sqrt{b}} \right\} \right]$$

$$\left[1 + \frac{K_2}{2}\sqrt{\frac{\pi}{b}}\ \frac{e^{-\frac{\omega^2}{4b}}}{v_2}\ \left(\mathrm{Re}\left\{\mathrm{erf}\left[\sqrt{b}\,(t+\rho) - \frac{j\omega}{2\sqrt{b}}\right]\right.\right.\right.$$

$$\left.\left.\left.- \mathrm{erf}\left[\sqrt{b}\,(t_2+\rho) - \frac{j\omega}{2\sqrt{b}}\right] + \frac{d_2\sqrt{b}}{u_2}\ \mathrm{erf}^1\left[\sqrt{b}\,t_2+\rho) - \frac{j\omega}{2\sqrt{b}}\right]\right\}\right)\right]^{-1}$$

$$\delta_2 = - \frac{F_2(t)}{1 + \frac{F_2(t)}{\phi_2} + G_2(t)} \quad , \tag{4.19}$$

where

$$F_2(t) = \frac{K_2 e^{-\frac{\omega^2}{4b}}}{2\,v_2}\sqrt{\frac{\pi}{b}}\ \phi_2\ \mathrm{Re}\left\{\mathrm{erf}\left[\sqrt{b}\,(t+\rho) - \frac{j\omega}{2\sqrt{b}}\right] - \mathrm{erf}\left[\sqrt{b}\,(t_2+\rho) - \frac{j\omega}{2\sqrt{b}}\right]\right\}$$

$$G_2(t) = \frac{K_2}{2v_2}\sqrt{\frac{\pi}{b}}\ e^{-\frac{\omega^2}{4b}}\ \frac{d_2\sqrt{b}}{v_2}\ \mathrm{erf}^1\left[\sqrt{b}\,(t_2+\rho) - \frac{j\omega}{2\sqrt{b}}\right] \quad .$$

Thus, the expression for δ_2 is very similar to that for δ_1.

As before, the total current induced as a result of the passage of charge in the interval Γ_2 is

$$i_3 = \int_{t-\Gamma_2}^{t} \frac{i_2}{d_2}\ u_2\ dt_2 \quad ,$$

where u_2 is the velocity at any instant t in gap 2; therefore

$$i_3 = \frac{1}{d_2}\int_{t-\Gamma_2}^{t} u_2\,dt_2 \left(\frac{i_1}{\frac{dt_2}{dt_1}}\right)$$

$$= \frac{I_o}{d_1 d_2} \int_{t-\Gamma_2}^{t} u_2 dt_2 \int_{t-\Gamma_1}^{t} \frac{u\, dt_o}{\frac{dt_2}{dt_1}} \qquad ,$$

where u is, as usual, the velocity at any instant t in the first gap. As we have seen in the above analysis dt_2/dt_1 can be expressed as a function of t_1 ; therefore

$$i_3 = \frac{I_o}{d_1 d_2} \frac{dt_1}{dt_2} \int_{t-\Gamma_2}^{t} u_2 dt_2 \int_{t-\Gamma_1}^{t} u\, dt_o \qquad .$$

This analysis can be extended to n gaps, and n-1 drift spaces; under these conditions

$$i_{2n-1} = \frac{I_o \left(\frac{dt_1}{dt_2} \cdot \frac{dt_3}{dt_4} \cdots \right)}{d_1 d_2 \cdots d_n} \int_{t-\Gamma_n}^{t} u_{2(n-1)}\, dt_{2(n-1)} \int_{t-\Gamma_{n-1}}^{t} u_{2(n-2)} dt_{2(n-2)} \int_{t-\Gamma_1}^{t} u\, dt_o \qquad . \tag{4.20}$$

Expression (4.20) is of theoretical interest since it indicates the dependence of the final current in the output gap upon the excitation imposed upon all other gaps.

V. NONLINEAR SPACE-CHARGE WAVE ANALYSIS

Here also attention will be directed to the Gaussian excitation, since the behavior of an electron passing through alternate gaps and drift regions with sinusoidal excitation in the gap has already been determined. McIsaac[3] has derived a general expression for polarization in the drift region. Using his symbols for the input gap and the drift case, the polarization $Z_1(T_1, T_o)$ in the drift region is:

$$Z_1(T_1, T_o) = -\frac{1}{2D}\int_0^{\Gamma} \theta(\Gamma + T_o)\sin(T_1 - \Gamma)\,d\Gamma \quad .$$

For the Gaussian case, $\theta(T) = Ae^{-\gamma T^2}\cos\sigma T$,

$$Z_1(T_1, T_o) = -\frac{1}{2D}\int_0^{\Gamma} Ae^{-\gamma(r+T_o)^2}\cos\sigma(r+T_o)\sin(T_1 - r)\,dr \quad .$$

Replacing the sine and cosine terms by exponentials and defining

$$\operatorname{erf} x = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}\,dt \quad ,$$

we have

$$Z(T, T_o) = \frac{A}{16}\sqrt{\frac{\pi}{\gamma}}\left(je^{\frac{(\sigma-1)^2}{4\gamma}} e^{jT}\left\{\operatorname{erf}\left[\sqrt{\gamma}(r+T_o) - j\frac{(\sigma-1)}{2\sqrt{\gamma}}\right] - \operatorname{erf}\left[\sqrt{\gamma}\,T_o - j\frac{(\sigma-1)}{2\sqrt{\gamma}}\right]\right\}\right.$$

$$-je^{\frac{(\sigma+1)^2}{4\gamma}} e^{-jT}\left\{\operatorname{erf}\left[\sqrt{\gamma}(r+T_o) - j\frac{(\sigma+1)}{2\sqrt{\gamma}}\right] - \operatorname{erf}\left[\sqrt{\gamma}\,T_o - j\frac{(\sigma+1)}{2\sqrt{\gamma}}\right]\right\}$$

$$\left. + \text{ complex conjugate}\right) \quad . \qquad (5.1)$$

For the input gap region,

$$Z_1(T_1 T_o) = -\frac{1}{2D} \int_0^{T_1} \theta(r + T_o) \sin(T_1 - r) \; dr \quad ;$$

therefore, for the case under consideration:

$$\begin{aligned} Z_1(T_1, T_o) = \frac{A}{16}\sqrt{\frac{\pi}{\gamma}} \Bigg(& je^{-\frac{(\sigma-1)^2}{4\gamma}} e^{jT} \left\{ \operatorname{erf}\left[\sqrt{\gamma}\, T - j\frac{(\sigma-1)}{2\sqrt{\gamma}}\right] - \operatorname{erf}\left[\sqrt{\gamma}\, T_o - j\frac{(\sigma-1)}{2\sqrt{\gamma}}\right]\right\} \\ & - je^{-\frac{(\sigma+1)^2}{4\gamma}} e^{-jT} \left\{ \operatorname{erf}\left[\sqrt{\gamma}\, T - \frac{(\sigma+1)}{2\sqrt{\gamma}}\right] - \operatorname{erf}\left[\sqrt{\gamma}\, T_o - j\frac{(\sigma+1)}{2\sqrt{\gamma}}\right]\right\} \\ & + \text{complex conjugate} \Bigg) \quad . \end{aligned} \tag{5.2}$$

A method will be indicated to express the complex error function in terms of real integrals. Consider the expression,

$$\operatorname{erf}(a-jb) = \frac{2}{\sqrt{\pi}} \int_0^{a-jb} e^{-z^2} \, dz \quad ,$$

in the complex plane with $z = x + jy$. As e^{-z^2} is an analytic function, $\oint f(z) = 0$ around a closed path of integration. Hence integrating along the path shown by the arrows gives

$$\int_0^a e^{-x^2} dx - j \int_0^b e^{-(a-jy)^2} dy = \int_0^{a-jb} e^{-z^2} dz \quad ,$$

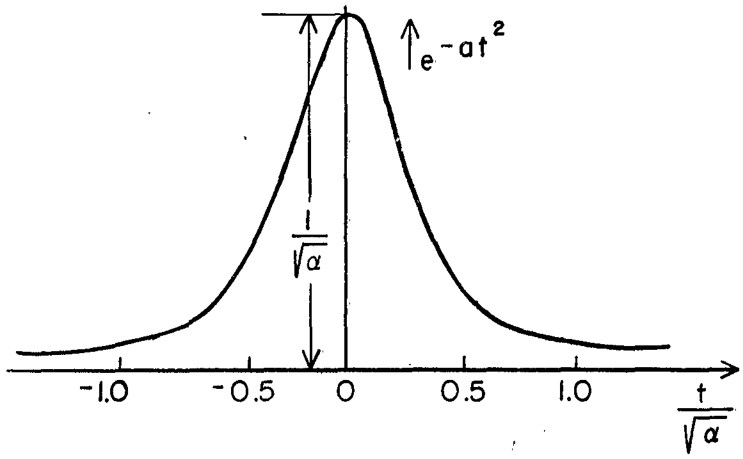


Figure 12. Gaussian Envelope.

from which

$$\operatorname{erf}(a-jb) = \frac{2}{\sqrt{\pi}}\int_0^\infty e^{-x^2}dx - \frac{2}{\sqrt{\pi}}j\int_0^b e^{-(a^2-y^2)+2jay}\,dy$$

$$= \operatorname{erf}(x) + \frac{2}{\sqrt{\pi}}\int_0^b e^{-(a^2-y^2)}\sin 2ay\,dy - j\frac{2}{\sqrt{\pi}}\int_0^b e^{-(a^2-y^2)}\cos 2ay\,dy \quad . \tag{5.3}$$

As the polarization has to be a real quantity, and noting that $T_o = T - T_1 = T - Z_o = T - Z + Z_1$, we have Equation (5.3) written in the Z, T coordinate system as follows:

$$Z_1(Z,T) = 2\mathrm{Re}\,\frac{A}{16}\sqrt{\frac{\pi}{\gamma}}\left(je^{-\frac{(\sigma-1)^2}{4\gamma}}e^{jT}\left\{\operatorname{erf}\left[\sqrt{\gamma}(r+T-Z+Z_1) - j\frac{(\sigma-1)}{2\sqrt{\gamma}}\right]\right.\right.$$

$$\left. - \operatorname{erf}\left[\sqrt{\gamma}(T - Z + Z_1) - j\frac{(\sigma-1)}{2\sqrt{\gamma}}\right]\right\}$$

$$- je^{-\frac{(\sigma+r)^2}{4\gamma}}e^{-jT}\left\{\operatorname{erf}\left[\sqrt{\gamma}(r+T-Z+Z_1) - j\frac{(\sigma+1)}{2\sqrt{\gamma}}\right]\right.$$

$$-\operatorname{erf}\left[\sqrt{\gamma}\,(T-Z+Z_1)\ -j\,\frac{(\sigma+1)}{2\sqrt{\gamma}}\right]\Bigg\}\Bigg) \qquad , \qquad (5.4)$$

where Re represents the real part. Equation (5.4) is obviously implicit in Z_1 and hence cannot be evaluated easily.

Assuming Z_1 to be small, a series expansion of the complex functions will be made, and only the first-order term in Z_1 will be taken into account. A typical term is expanded as follows:

$$\operatorname{erf}\left[\sqrt{\gamma}\,(r+T-Z+Z_1)\ -j\,\frac{(\sigma-1)}{2\sqrt{\gamma}}\right]$$

$$= \operatorname{erf}\left[\sqrt{\gamma}\,(r+T-Z)-j\,\frac{(\sigma-1)}{2\sqrt{\gamma}}\right] + \sqrt{\gamma}\,Z_1\,\operatorname{erf}^1\left[\sqrt{\gamma}\,(r+T-Z)\ -j\,\frac{(\sigma-1)}{2\sqrt{\gamma}}\right] \qquad ,$$

where

$$\operatorname{erf}^1 = e^{-\left[\sqrt{\gamma}\,(r+T-Z)\ -j\,\frac{(\sigma-1)}{2\sqrt{\gamma}}\right]^2} \quad .$$

Substituting in Equation (5.4), we have

$$Z_1(Z,T) = \frac{A}{8}\sqrt{\frac{\pi}{\gamma}}\ \mathrm{Re}\left[j e^{-\frac{(\sigma-1)^2}{4\gamma}}\ e^{jT}\left(\left\{\operatorname{erf}\left[\sqrt{\gamma}(r+T-Z)\ -j\,\frac{(\sigma-1)}{2\sqrt{\gamma}}\right]\right.\right.\right.$$

$$\left. + \sqrt{\gamma}\,Z_1\ e^{-\left[\sqrt{\gamma}\,(r+T-Z)\ -j\,\frac{(\sigma-1)}{2\sqrt{\gamma}}\right]^2}\right\} - \left\{\operatorname{erf}\left[\sqrt{\gamma}\,(T-Z)-j\,\frac{(\sigma-1)}{2\sqrt{\gamma}}\right]\right.$$

$$\left.\left. + \sqrt{\gamma}\,Z_1 e^{-\left[\sqrt{\gamma}\,(T-Z)-j\,\frac{(\sigma-1)}{2\sqrt{\gamma}}\right]^2}\right\}\right) - j e^{-\frac{(\sigma+1)^2}{4\gamma}}\ e^{-jT}\left(\left\{\operatorname{erf}\left[\sqrt{\gamma}\,(r+T-Z)\right.\right.\right.$$

$$-j\,\frac{(\sigma+1)}{2\sqrt{\gamma}}\Bigg] + \sqrt{\gamma}\, Z_1\, e^{-\left[\sqrt{\gamma}\,(r+T-Z) - j\,\frac{(\sigma+1)}{2\sqrt{\gamma}}\right]^2}\Bigg\} - \Bigg\{ \mathrm{erf}\Bigg[\sqrt{\gamma}\,(T-Z)$$

$$-j\,\frac{(\sigma+1)}{2\sqrt{\gamma}}\Bigg] + \sqrt{\gamma}\, Z_1\, e^{-\left[\sqrt{\gamma}\,(T-Z) - j\,\frac{(\sigma+1)}{2\sqrt{\gamma}}\right]^2}\Bigg\}\Bigg)\Bigg] \quad . \qquad (5.5)$$

Now, putting

$$\sqrt{\gamma}\,(r+T-Z) = a_1 \quad , \qquad \sqrt{\gamma}\,(T-Z) = a_2$$

$$\frac{(\sigma-1)}{2\sqrt{\gamma}} = b_1 \quad , \qquad \frac{\sigma+1}{2\sqrt{\gamma}} = b_2 \quad ,$$

in Equation (5.5), taking the real parts, and transferring Z_1 to the left-hand side, one derives an explicit expression for Z_1 as follows:

$$Z_1(Z, T) = \frac{A}{8}\sqrt{\frac{\pi}{\gamma}}\Bigg[e^{-\frac{(\sigma-1)^2}{4\gamma}} \Bigg(\Bigg\{ \mathrm{erf}(a_2) - \mathrm{erf}(a_1) + \frac{2}{\sqrt{\pi}} \int_0^{b_1} \Big[e^{-\left(a_2^2 - y^2\right)} \sin 2a_2 y$$

$$- e^{-\left(a_1^2 y^2\right)} \sin 2a_1 y \Big] dy \Bigg\} \sin T + \Bigg\{ \frac{2}{\sqrt{\pi}} \int_0^{b_1} \Big[e^{-\left(a_1^2 - y^2\right)} \cos 2a_1 y$$

$$- e^{-\left(a_2^2 - y^2\right)} \cos 2a_2 y \Big] dy \Bigg\} \cos T \Bigg) + e^{-\frac{(\sigma+1)^2}{4\gamma}} \Bigg(\Bigg\{ \mathrm{erf}(a_2) - \mathrm{erf}(a_1)$$

$$+ \frac{2}{\sqrt{\pi}} \int_0^{b_2} \Big[e^{-\left(a_2^2 - y^2\right)} \sin 2a_2 y - e^{-\left(a_1^2 - y^2\right)} \sin 2a_1 y \Big] dy \Bigg\} \sin T$$

$$+\left[\frac{2}{\sqrt{\pi}}\int_0^{b_2} e^{-\left(a_2^2-y^2\right)}\cos 2a_2y\ dy - e^{-\left(a_1^2-y^2\right)}\cos 2a_1y\ dy\right]\cos T\Bigg)\Bigg]$$

$$\Bigg(1+\frac{A}{8}\sqrt{\pi}\left\{\left[e^{-\left(a_1^2-b_1^2\right)}\cos 2a_1b_1 - e^{-\left(a_2^2-b_1^2\right)}\cos 2a_2b_1\right]\sin T\right.$$

$$\left.+\left[e^{-\left(a_1^2-b_1^2\right)}\sin 2a_1b_1 - e^{-\left(a_2^2-b_1^2\right)}\sin 2a_2b_1\right]\cos T\right\} e^{-\frac{(\sigma-1)^2}{4\gamma}}$$

$$+\frac{A}{8}\sqrt{\pi}\left\{\left[e^{-\left(a_1^2-b_2^2\right)}\cos 2a_1b_2 - e^{-\left(a_2^2-b_2^2\right)}\cos 2a_2b_2\right]\sin T\right.$$

$$\left.-\left[e^{-\left(a_1^2-b_2^2\right)}\sin 2a_1b_2 - e^{-\left(a_2^2-b_2^2\right)}\sin 2a_2b_2\right]\cos T\right\} e^{-\frac{(\sigma+1)^2}{4\gamma}}\Bigg)^{-1} .$$

If

$$\frac{2}{\sqrt{\pi}}\int_0^{b_n} e^{-\left(a_m^2-y^2\right)}\sin 2a_my\ dy = F(a_m, b_n)$$

$$\frac{2}{\sqrt{\pi}}\int_0^{b_n} e^{-\left(a_m^2-y^2\right)}\cos 2a_my\ dy = f(a_m, b_n) \quad ;$$

then

$$Z_1(Z, T) = \frac{A}{8}\sqrt{\frac{\pi}{\gamma}}\left(e^{-\frac{(\sigma-1)^2}{4\gamma}}\left\{\left[\mathrm{erf}(a_2)-\mathrm{erf}(a_1)+F(a_2, b_1)-F(a_1, b_1)\right]\sin T\right.\right.$$

$$\left.+\left[f(a_1, b_1) - f(a_2b_1)\right]\cos T\right\}$$

$$+e^{-\frac{(\sigma+1)^2}{4\gamma}}\left\{\left[\mathrm{erf}(a_2)-\mathrm{erf}(a_1)+F(a_2, b_2)-F(a_1, b_2)\right]\sin T\right.$$

$$+\left[f(a_2,b_2) - f(a_1,b_2)\right] \cos T\Big\}\Big)$$

$$\Bigg(1 + \frac{A}{8}\sqrt{\pi}\Bigg\{\left[e^{-\left(a_1^2-b_1^2\right)}\cos 2a_1b_1 - e^{-\left(a_2^2-b_1^2\right)}\cos 2a_2b_1\right] \sin T$$

$$+\left[e^{-\left(a_1^2-b_1^2\right)}\sin 2a_1b_1 - e^{-\left(a_2^2-b_2^2\right)}\sin 2a_2b_2\right] \cos T\Bigg\} e^{-\frac{(\sigma-1)^2}{4\gamma}}$$

$$+\frac{A}{8}\sqrt{\pi}\Bigg\{\left[e^{-\left(a_1^2-b_2^2\right)}\cos 2a_1b_2 - e^{-\left(a_2^2-b_2^2\right)}\cos 2a_2b_2\right] \sin T$$

$$-\left[e^{-\left(a_1^2 b_2^2\right)}\sin 2a_1b_2 - e^{-\left(a_2^2-b_2^2\right)}\sin 2a_2b_2\right] \cos T\Bigg\} e^{-\frac{(\sigma+1)^2}{4\gamma}}\Bigg)^{-1} \quad . \tag{5.6}$$

Now putting

$$\sqrt{\gamma}\,(r + T - Z + Z_1) = a_3$$

$$\sqrt{\gamma}\,(T - Z + Z_1) = a_4$$

$$e^{-\left(a_m^2-b_n^2\right)}\sin 2a_mb_n = G(a_m, b_n)$$

$$e^{-\left(a_m^2-b_n^2\right)}\cos 2a_mb_n = g(a_m, b_n)$$

$$\operatorname{erf}(a_m) - \operatorname{erf} a_n = E(a_m-a_n) \quad ,$$

we obtain

$$\frac{\partial Z_1}{\partial T}(Z,T) = \frac{A}{8}\sqrt{\frac{\pi}{\gamma}}\Bigg[e^{-b_1^2}\Bigg(\Big\{\sqrt{\gamma}\left[g(a_4,b_1)-g(a_3,b_1)\right] + f(a_3,b_1)-f(a_4,b_1)\Big\} \sin T$$

$$+\Big\{\sqrt{\gamma}\left[G(a_4,b_1)-G(a_3,b_1)\right] - E(a_3-a_4) + F(a_4,b_1) - F(a_3,b_1)\Big\} \cos T\Bigg)$$

$$-e^{-b_2^2}\Bigg(\Big\{\sqrt{\gamma}\left[g(a_3,b_2)-g(a_4,b_2)\right] + f(a_4,b_2)-f(a_3,b_2)\Big\} \sin T$$

$$+\left\{\sqrt{\gamma}\left[G(a_4,b_2)-G(a_3,b_2)\right]+E(a_3-a_4)+F(a_3,b_2)-F(a_4,b_2)\right\}\cos T\Big)\Bigg]$$

$$1+\frac{A}{8}\sqrt{\pi}\Bigg(e^{-b_1^2}\left\{\left[g(a_3,b_1)-g(a_4,b_1)\right]\sin T+\left[G(a_3,b_1)-G(a_4,b_1)\right]\cos T\right\}$$

$$+e^{-b_2^2}\left\{\left[g(a_3,b_2)-g(a_4,b_2)\right]\sin T+\left[G(a_4,b_2)-G(a_3,b_2)\right]\cos T\right\}\Bigg)^{-1} . \tag{5.7}$$

In Equation (5.7), the term Z_1 occurs on the right-hand side. To obtain $\partial Z_1/\partial T$ at a particular Z and T, therefore, the value of Z_1 has to be obtained from Equation (5.6) for the given Z, T, and then has to be substituted in Equation (5.7). Thus, the ratio J_{a-c}/J_o can be calculated. Now, from Equation (5.5)

$$\frac{\partial Z_1}{\partial Z}=\frac{A}{8}\sqrt{\frac{\pi}{\gamma}}\Bigg(e^{-b_1^2}\left\{\left[g(a_3,b_1)-g(a_4,b_1)\right]\sin T+\left[-G(a_4,b_1)+G(a_3,b_1)\right]\cos T\right\}\sqrt{\gamma}$$

$$e^{-b_2^2}\left\{\left[g(a_4,b_2)-g(a_3,b_2)\right]\sin T+\left[G(a_3,b_2)-G(a_4,b_2)\right]\cos T\right\}\sqrt{\gamma}\Bigg)$$

$$\Bigg[1+\frac{A}{8}\sqrt{\frac{\pi}{\gamma}}\Bigg(e^{-b_1^2}\left\{\left[g(a_3,b_1)-g(a_4,b_1)\right]\sin T+\left[G(a_4,b_1)-G(a_3,b_1)\right]\cos T\right\}\sqrt{\gamma}$$

$$+e^{-b_2^2}\left\{\left[-g(a_4,b_2)+g(a_3,b_2)\right]\sin T+\left[-G(a_3,b_2)+G(a_4,b_2)\right]\cos T\right\}\sqrt{\gamma}\Bigg)\Bigg]^{-1} . \tag{5.8}$$

Since

$$\frac{dZ_1}{dT} \doteq \frac{\dfrac{\partial Z_1}{\partial Z}+\dfrac{\partial Z_1}{\partial T}}{1-\dfrac{\partial Z_1}{\partial Z}} ;$$

then

$$\frac{dZ_1}{dT} = \frac{A}{8}\sqrt{\frac{\pi}{\gamma}}\left(e^{-b_1^2}\left\{\left[f(a_3,b_1)-f(a_4,b_1)\right]\sin T+\left[-E(a_3-a_4)+F(a_4,b_1)-F(a_3,b_1)\right]\cos T\right\}\right.$$

$$\left.-e^{-b_2^2}\left\{\left[f(a_4,b_2)-f(a_3,b_2)\right]\sin T+\left[E(a_3-a_4)+F(a_3,b_2)-F(a_4,b_2)\right]\cos T\right\}\right) \quad . \tag{5.9}$$

The relative simplicity in the expression for

$$\frac{u}{u_o} = \frac{\partial Z_1}{\partial T}$$

is observed.

VI. CONCLUSIONS AND RECOMMENDATIONS

The ballistic theory of an electron beam has been developed, for multiple signals, subject to approximations made to simpfly solution. Some work, using nonlinear space-charge wave analysis, has also been done and a comparison of the ballistic and space-charge wave analysis should be attempted to throw more light on the problem. The graphic plot obtained (Figure 9) which gives a value of 0.5 for the bunching parameter indicates that there is pulse distortion. The saturation of the lower half of the envelope indicates that the exit current is rich in harmonics, even for low depths of modulation.

The theoretical study indicates that experiments can provide solutions where theoretical formulation would be cumbersome. An experimental verification of the theory should therefore be undertaken, using a set of parameters designed to approximate closely the assumption of an infinite beam and no space-charge effects.

Attention is now being directed to the generation of high peak-power radar using nanosecond pulses of the Gaussian type. Once such pulses are generated, the response of the klystron to them can be observed. Good microwave amplification of these pulses would lead to their use in radars.

APPENDIX A: A NOTE ON THE GAUSSIAN SPECTRUM

As this report deals with the klystron response to a Gaussian envelope, a short note on the Gaussian spectrum is useful. The analysis presented has been mostly based on the envelope $g(t) = Ve^{-at^2} \cos \omega t$:

$$g(\omega_o) = \mathrm{Re}\, \frac{1}{2\pi} \int_{-a}^{a} Ve^{-at^2} e^{-j\omega_o t} e^{+j\omega t}\, dt$$

$$= \mathrm{Re}\, \frac{1}{2\pi} \int_{-a}^{a} Ve^{-a\left(t^2 + j\frac{(\omega_o - \omega)}{a}\right)} dt$$

$$= Ve^{-\frac{(\omega_o - \omega)^2}{4a}} \frac{1}{2\pi} \int_{-a}^{a} e^{-a\left(t + j\frac{(\omega_o - \omega)}{2a}\right)^2} dt$$

$$= \frac{1}{2\pi} \sqrt{\frac{\pi}{a}}\, Ve^{-\frac{(\omega_o - \omega)^2}{4a}} .$$

Thus, in the frequency plane the envelope is also Gaussian. The Gaussian envelope is evidently economical in bandwidth for a given pulse length, the majority of the energy being confined to a finite range of the frequency spectrum centered on the carrier frequency. These factors together with the fact that a Gaussian pulse is easier to generate, are the criterion determining its selection for analysis. It might be interesting to note to what extent the envelope shape depends on the parameter a. Consider just the envelope given by the equation,

$$f(t) = e^{-at^2} \quad ;$$

then

$$g(\omega_o) = \frac{1}{2} \frac{e^{-\frac{\omega_o^2}{4a}}}{\sqrt{\pi a}} .$$

Select $a = \pi/\alpha$, and multiply both $f(t)$ and $f(\omega_o)$ by $1/\sqrt{\alpha}$; then

$$f_1(t) = \frac{e^{-\frac{\pi t^2}{\alpha}}}{\sqrt{\alpha}} ,$$

$$g_1(\omega_o) = \frac{e^{-\frac{\alpha\omega_o^2}{4\pi}}}{2\pi} .$$

The area under the curve $f_1(t) = e^{-at^2}/\sqrt{\alpha}$ is given by

$$A = 2\pi \, g_1(\omega_o)^1 \Big|_{\omega_o = 0}$$

$$= \frac{2\pi}{2\pi} = 1 .$$

Therefore as α becomes smaller and smaller, the curve $f_1(t)$ becomes taller and narrower and approaches a unit impulse as α approaches zero. Since a is inversely proportional to α, one must have a high value of a to obtain short pulses. If the frequency is increased, the value of a has to be increased also, if the pulse is to decay to a fixed fraction of its amplitude after a fixed number of r-f cycles. Actually for this purpose the ratio a/ω^2, where $\omega = 2\pi x$ frequency, has to be maintained constant.

The transition from a frequency spectrum consisting of a series of

discrete frequencies to one consisting of a continuous band of frequencies can be made by treating the nonperiodic function as a periodic function in which the period approaches α . The unit Gaussian envelope $e^{-at^2} \cos \omega t$ will be considered. The amplitude of the spectrum at ω_o, is

$$g(\omega_o) = \frac{1}{2\pi} \operatorname{Re} \int_{-\alpha}^{\alpha} e^{-at^2 + j\omega t} e^{-j\omega_o t} dt \quad .$$

For a single pulse, where $f(t) = 0$ for all values of t except $-L < t < L$, we have

$$g(\omega_o) = \frac{1}{2\pi} e^{-\frac{(\omega - \omega_o)^2}{4a}} \operatorname{Re} \int_{-L}^{L} e^{-a\left[t - j\frac{(\omega - \omega_o)}{2a}\right]^2} dt$$

$$= \frac{1}{2\pi\sqrt{a}} e^{-\frac{(\omega - \omega_o)^2}{4a}} \operatorname{Re}\left[\operatorname{erf}\sqrt{a}\left(L - j\frac{(\omega - \omega_o)}{2a}\right) + \operatorname{erf}\sqrt{a}\left(L + j\frac{(\omega - \omega_o)}{2a}\right)\right] ,$$

where use has been made of the identity,

$$\operatorname{erf}(-x) = -\operatorname{erf}(x) \quad .$$

With the expansion for the complex error function and then taking the real part, one obtains

$$g(\omega_o) = \frac{1}{2\pi\sqrt{a}} e^{-\frac{(\omega - \omega_o)^2}{4a}} (2 \operatorname{erf}\sqrt{a}\, L)$$

$$= \left(\frac{\operatorname{erf}\sqrt{a}\, L}{\pi\sqrt{a}}\right) e^{-\frac{(\omega - \omega_o)^2}{4a}} \quad .$$

The spectrum of a train of Gaussian pulses of length $2(L + \Delta L)$ recurring every T seconds will be found from the spectrum of a single pulse of the train. For the single pulse at any frequency $\omega_o/2\pi$,

$$g(\omega_o) = \frac{\operatorname{erf}\sqrt{a}\,(L + \Delta L)}{\pi\sqrt{a}} e^{-\frac{(\omega - \omega_o)^2}{4a}} .$$

For a period of such pulses recurring with a spacing $T = 1/C$, the sum of spectra of the individual pulses form a Fourier series of harmonics of C; therefore

$$f(t) = A_o + \sum_{n=1}^{a} A_n \cos 2\pi nCT ,$$

where A_n is the sum of an infinite number (one from each pulse) of infinitesimal terms $g(2\pi nC)$ and $g(-2\pi nC)$, giving

$$A_n = \sum \frac{\operatorname{erf}\sqrt{a}\,(L + \Delta L)}{\pi\sqrt{a}} e^{-\frac{(\omega' - 2\pi nC)^2}{4a}} .$$

To put an absolute value on the amplitudes $g(\omega_o)$, it is necessary to average them over the recurrence period of the single pulse, making them infinitesimals. However, in the train of pulses recurring every $T = 1/C$ seconds, the amplitude of A_n can be determined by averaging the terms in $g(\omega_o)$ over an interval T; then

$$A_n = \frac{\operatorname{erf}\sqrt{a}\,(L + \Delta L)}{\pi\sqrt{a}\;T} e^{-\frac{(\omega - 2\pi nC)^2}{4a}} ,$$

and when $T = 4L \mp \frac{1}{C}$,

$$A_n = \frac{C \operatorname{erf} \sqrt{a}\ L \left(1 + \frac{\Delta L}{L}\right)}{\pi \sqrt{a}\ (4LC \mp 1)} \quad - \frac{(\omega - 2\pi nC)^2}{4a} \quad ;$$

thus,

$$f(t) = A_o + A_1 \cos 2\pi Ct + A_2 \cos 2\pi 2Ct + \ldots \quad ,$$

where A_n is known.

APPENDIX B. FOURIER COEFFICIENTS

The Fourier coefficients a_r and b_r, as obtained in Equation (2.9) are simplified here:

$$a_r = \frac{1}{\pi} \int_{-\pi}^{\pi} I_o \cos r \left[y - k_1' \sin\left(my + \frac{\phi_1'}{2}\right) - k_1'' \sin\left(ny + \frac{\phi_1''}{2}\right)\right] dy \quad .$$

It is sufficient to obtain the solutions to the following coefficients, as they are related to the ones in Equation (2.9):

$$a_r' = \int_{-\pi}^{\pi} \cos r \left[y + c_1 \sin(my + \alpha_1) + c_2 \sin(ny + \alpha_2)\right] dy \quad ,$$

$$b_r' = \int_{-\pi}^{\pi} \sin r \left[y + c_1 \sin(my + \alpha_1) + c_2 \sin(ny + \alpha_2)\right] dy \quad ,$$

$$a_r' + j b_r' = \int_{-\pi}^{\pi} e^{jr\left[y + c_1 \sin(my + \alpha_1) + c_2 \sin(ny + \alpha_2)\right]} dy \quad ,$$

$$a_r' - j b_r' = \int_{-\pi}^{\pi} e^{-jr\left[y + c_1 \sin(my + \alpha_1) + c_2 \sin(ny + \alpha_2)\right]} dy \quad .$$

Now, according to the property of Bessel Functions,

$$\begin{aligned} e^{jz\sin\theta} &= J_o(z) + 2\left[J_2(z) \cos 2\theta + J_4(z) \cos 4\theta + \ldots\right. \\ &\quad + 2j\left[J_1(z) \sin\theta + J_3(z) \sin 3\theta + \ldots\right. \\ &= \sum_{p=-\alpha}^{\alpha} Jp(z)\, e^{jp\theta} \quad , \end{aligned} \tag{B.1}$$

since $J_{-p}(z) = (1)^{P} J_p(z)$; therefore, using the property in Equation (B. 1) gives

$$a_r' + j b_r' = \int_{-\pi}^{\pi} e^{jry} \sum_{p=-a}^{a} J_p(rc_1) \quad e^{jp(my + a_1)} \sum_{q=-a}^{a} J_q(rc_2) \; e^{jq(ny + a_2)} \; dy$$

$$= \int_{-\pi}^{\pi} e^{jry} \sum_{p=-a}^{a} \sum_{q=-a}^{a} J_p(rc_1) \; Jq(rc_2) \; e^{j(pm+qn)y + j(pa_1 + qa_2)} \; dy \quad .$$

Because of the nature of the integrand, the order of integration and summation can be interchanged; therefore

$$a_r' + jb_r' = 2j \sum_{p=-a}^{a} \sum_{q=-a}^{a} J_p(rc_1) \; J_q(rc_2) \; e^{j(pa_1 + qa_2)} \; \frac{\sin(pm+qn+r)}{pm+qn+r} \; \pi \quad .$$

Now, noting that

$$J_v(-z) = J_v(ze^{j\pi}) = \frac{\sum_{m=o}^{a} (-1)^m \left(\frac{1}{2} \; z \, e^{j\pi}\right)^{v+2m}}{m! \; \Gamma(v+m+1)} = e^{jv\pi} \sum_{m=o}^{a} \frac{(-1)^m \left(\frac{1}{2} z\right)^{v+2m}}{m! \; \Gamma(v+m+1)} \quad ,$$

since $e^{j 2\pi m} = 1$, therefore

$$J_v(-z) = e^{jv\pi} \; J_v(z) \quad .$$

Using the same procedure as before, we obtain

$$a_r' - jb_r' = 2j \sum_{p=-a}^{a} \sum_{q=-a}^{a} J_p(rc_1) \, J_q(rc_2) \; e^{jp(\pi + a_1) + jq(\pi + a_2)} \; \frac{\sin(pm+qn-r)\pi}{(pm+qn-r)}$$

$$= 2j \sum_{p=-\alpha}^{\alpha} \sum_{q=-\alpha}^{\alpha} (-1)^{p+q} J_p(rc_1) J_q(rc_2) e^{j(p\alpha_1 + q\alpha_2)} \frac{\sin(pm + qn - r)\pi}{pm + qn - r},$$

from which

$$a_r' = 2j \sum_{p=-\alpha}^{\alpha} \sum_{q=-\alpha}^{\alpha} J_p(rc_1) J_q(rc_2) e^{j(p\alpha_1 + q\alpha_2)} \cdot$$

$$\frac{(pm + qn - r) \sin(pm + qn + r)\pi + (-1)^{p+q} (pm + qn + r) \sin(pm + q - r)\pi}{(pm + qn)^2 - r^2},$$

$$b_r' = 2 \sum_{p=-\alpha}^{\alpha} \sum_{q=-\alpha}^{\alpha} J_p(rc_1) J_q(rc_2) e^{j(p\alpha_1 + q\alpha_2)}$$

$$\frac{(pm + qn - r) \sin(pm + qn + r)\pi - (-1)^{p+q} (pm + qn + r) \sin(pm + qn - r)\pi}{(pm + qn)^2 - r^2}.$$

As r, p, and q are integers, a_r', and b_r' will not be zero when m and n do not have integral values. It must be noted that for large values of p and q, the quantity $1/\left[(pm + qn)^2 - r^2\right]$ becomes small, and therefore the double infinite series can be replaced by a finite series to permit computation.

APPENDIX C. EVALUATION OF erf(a - jb)

In the analysis of klystron response to Gaussian wave excitation, complex error functions of the type erf(a - jb), where a and b are real numbers, have often been encountered. Here, a method will be indicated to express the complex error function in terms of real integrals. Consider the expression,

$$\operatorname{erf}(a - jb) = \frac{2}{\sqrt{\pi}} \int_0^{a-jb} e^{-z^2}\, dz \quad ,$$

in the complex plane with $z = x + jy$.

As e^{-z^2} is an analytic function, $\oint f(z)\, dz = 0$ around a closed path of integration. Hence, integrating along the path shown by the arrows in Figure 13 gives

$$\int_0^a e^{-x^2}\, dx - j \int_0^b e^{-(a-jy)^2}\, dy = \int_0^{a-jb} e^{-z^2}\, dz \quad ;$$

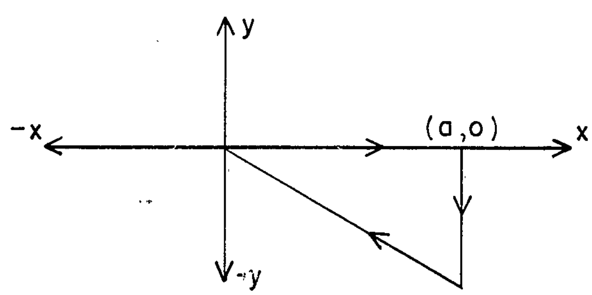


Figure 13. Path of Integration for the Function e^{-z^2}.

from which

$$\operatorname{erf}(a - jb) = \frac{2}{\sqrt{\pi}} \int_0^a e^{-x^2}\, dx - \frac{2}{\sqrt{\pi}} \int_0^b e^{-(a^2 - y^2) + 2jay}\, dy$$

$$= \operatorname{erf}(a) + \frac{2}{\sqrt{\pi}} \int_0^b e^{-(a^2 - y^2)} \sin 2ay\ dy - j\frac{2}{\sqrt{\pi}} \int_0^b e^{-(a^2 - y^2)} \cos 2ay\, dy\ .$$

This representation of a complex error function in terms of real integrals has been used frequently.

Some other important results follow:

$$\operatorname{erf}(-a - jb) = \operatorname{erf}(-a) - \frac{2}{\sqrt{\pi}} \int_0^b e^{-(a^2 - y^2)} \sin 2ay\ dy - \frac{j2}{\sqrt{\pi}} \int_0^b e^{-(a^2 - y^2)} \cos 2ay\ dy\ ,$$

as

$$\operatorname{erf}(a) = -\operatorname{erf}(-a)$$

$$\operatorname{erf}(a-jb) - \operatorname{erf}(-a - jb) = \frac{2}{\sqrt{\pi}} \int_{-a-jb}^{a-jb} e^{-z^2}\ dz$$

$$= 2\operatorname{erf}(a) + \frac{4}{\sqrt{\pi}} \int_0^b e^{-(a^2 - y^2)} \sin 2ay\ dy \quad ,$$

which is a real quantity. This same result can be obtained by contour integration around a suitable rectangle in the z-plane. Again

$$\operatorname{erf}(a - jb) - \operatorname{erf}(c - jb) = \operatorname{erf}(a) - \operatorname{erf}(c) + \frac{2}{\sqrt{\pi}} \int_0^b \left[e^{-(a^2 - y^2)} \sin 2ay \right.$$

$$\left. - e^{-(c^2 - y^2)} \sin 2cy \right] dy + j \frac{2}{\sqrt{\pi}} \int_0^b \left[e^{-(c^2 - y^2)} \cos 2cy \right.$$

$$\left. - e^{-(a^2 - y^2)} \cos 2ay \right] dy \quad .$$

These results show that a complex error function can be easily computed, and its real and imaginary parts separated. Considerable simplification in computation can result in specific problems. The asymptotic expansion, for example,

$$\operatorname{erf}(a) = 1 - \frac{e^{-a^2}}{a\sqrt{\pi}} \left[1 - \frac{1}{(2a^2)} + \frac{1 \cdot 3}{(2a^2)^2} - \frac{1 \cdot 3 \cdot 5}{(2a^2)^3} + \ldots \right] ,$$

is convenient for computation when a is large.

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